# Differentiation from 1st Principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

note: $\lim_{h \to 0} h = 0$

Course Requires : $x^2$, $x^3$, $\frac{1}{x}$, $\sqrt{x}$, $\sin x$, $\cos x$

Differentiate $x^2$ from first principles.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x) = x^2$

$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$

$$\frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2}{h} = 2x + h$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) = 2x$$

## Differentiate $x^3$ from first principles.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = x^3$$

$$f(x+h) = (x+h)^3 = (x+h)(x+h)^2 = (x+h)(x^2+2hx+h^2)$$

$$= x^3 + 2hx^2 + h^2x + hx^2 + 2h^2x + h^3$$

$$= x^3 + 3hx^2 + 3h^2x + h^3$$

$$\frac{f(x+h) - f(x)}{h} = \frac{3hx^2 + 3h^2x + h^3}{h} = 3x^2 + 3hx + h^2$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) = 3x^2$$

## Differentiate $\frac{1}{x}$ from first principles.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = \frac{1}{x}$$

$$f(x+h) = \frac{1}{x+h}$$

$$f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x} = \frac{x - (x+h)}{x(x+h)} = \frac{-h}{x^2+h^2}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\left(\frac{-h}{x^2+h^2}\right)}{h} = \frac{-1}{x^2+h^2}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) = \frac{-1}{x^2} \quad (\text{or } -x^{-2})$$

Differentiate $\sqrt{x}$ from first principles.

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

$$f(x) = \sqrt{x}$$

$$f(x+h) = \sqrt{x+h}$$

$$f(x+h) - f(x) = \sqrt{x+h} - \sqrt{x} = \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{(\sqrt{x+h} + \sqrt{x})} = \frac{x+h-x}{(\sqrt{x+h} + \sqrt{x})} = \frac{h}{(\sqrt{x+h} + \sqrt{x})}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{h}{(\sqrt{x+h} + \sqrt{x})}}{h} = \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \quad \left(= \frac{1}{2} x^{-\frac{1}{2}}\right)$$

# Limits of a trigonometric function

note:

$$\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \approx 1}$$

# Differentiate sinx from first principles

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

$$f(x) = \sin(x)$$
$$f(x+h) = \sin(x+h)$$

From log tables

$$\boxed{\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}$$

$$f(x+h) - f(x) = \sin(x+h) - \sin(x)$$

$$= 2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right) = 2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)$$

$$\frac{f(x+h)-f(x)}{h} = \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} = 2\cos\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{2\left(\frac{h}{2}\right)}$$

$$\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \approx 1}$$

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = f'(x) = \cancel{2}\cos\left(\frac{\cancel{2}x+0}{\cancel{2}}\right)\frac{1}{\cancel{2}}(1) = \cos x$$

# Differentiate cosx from first principles

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

$$f(x) = \cos(x)$$
$$f(x+h) = \cos(x+1)$$

From log tables

$$\boxed{\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}$$

$$f(x+h) - f(x) = \cos(x+h) - \cos(x)$$

$$= -2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right) = -2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)$$

$$\frac{f(x+h)-f(x)}{h} = \frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} = -2\sin\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{2\left(\frac{h}{2}\right)}$$

$$\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \approx 1}$$

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = f'(x) = -\cancel{2}\sin\left(\frac{\cancel{2}x+0}{\cancel{2}}\right)\frac{1}{\cancel{2}}(1) = -\sin x$$

HW 1-10-2012

Differentiate $2x^2 + 3x$ from 1st principles.

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

$$f(x) = 2x^2 + 3x$$

$$f(x+h) = 2(x+h)^2 + 3(x+h)$$

$$= 2(x^2 + 2xh + h^2) + 3x + 3h$$

$$= 2x^2 + 4xh + 2h^2 + 3x + 3h$$

$$= 2x^2 + (4h+3)x + (2h^2 + 3h)$$

$$\frac{f(x+h) - f(x)}{h} = \frac{4hx + 2h^2 + 3h}{h} = 4x^2 + 2h + 3$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) = 4x^2 + 3$$