Question 3 **(25 marks)**

(a) Prove, by induction, that the sum of the first n natural numbers,
$1+2+3+\cdots+n,$ is $\frac{n(n+1)}{2}.$

(b) Hence, or otherwise, prove that the sum of the first n even natural numbers,
$2+4+6+\cdots+2n,$ is $n^2+n.$

(c) Using the results from **(a)** and **(b)** above, find an expression for the sum of the first n odd natural numbers in its simplest form.

Question 3 **(25 marks)**

(a) Prove, by induction, that the sum of the first n natural numbers, $1+2+3+\cdots+n$, is $\frac{n(n+1)}{2}$.

To Prove: $P(n) = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$

$P(1)$: $1 = \frac{1(1+1)}{2} = 1$, True

Assume P(n) is true for n = k, and prove P(n) is true for n = k + 1.

n = k: $1+2+3+\cdots+k = \frac{k(k+1)}{2}$

To prove P(k+1) $= \frac{(k+1)}{2}(k+2)$

L.H.S. $= 1+2+3+\cdots+k+(k+1) = \frac{k(k+1)}{2}+(k+1) = \frac{k(k+1)}{2}+\frac{2(k+1)}{2}$

$= \frac{(k+1)}{2}(k+2) =$ R.H.S

But P(1) is true, so P(2) is true etc.
Hence, P(n) is true for all n.

(b) Hence, or otherwise, prove that the sum of the first n even natural numbers, $2+4+6+\cdots+2n$, is n^2+n.

a = 2 and d = 2.

$S_n = \frac{n}{2}\left(2a+(n-1)d\right) = \frac{n}{2}\left(4+(n-1)2\right) = \frac{n}{2}\left(2n+2\right) = n^2+n$

OR

$S_n = 2+4+6+\ldots\ldots\ldots\ldots+2n$

$= 2(1+2+3+\ldots\ldots\ldots\ldots+n)$

$= 2\left[\frac{n(n+1)}{2}\right]$

$= n(n+1)$

$= n^2+n$

(c) Using the results from **(a)** and **(b)** above, find an expression for the sum of the first n odd natural numbers in its simplest form.

$$1+2+3+\cdots+2n=\frac{2n(2n+1)}{2}=2n^2+n$$

$$\Rightarrow (1+3+5+\cdots n \text{ terms})+(2+4+6+\cdots n \text{ terms})=2n^2+n$$

$$\Rightarrow (1+3+5+\cdots n \text{ terms})+(n^2+n)=2n^2+n$$

$$\Rightarrow 1+3+5+\cdots n \text{ terms } =2n^2+n-(n^2+n)=n^2$$

OR

$$S_A=1+2+3+\dots\dots\dots\dots\dots+(2n-1)+(2n) \qquad =2n^2+n$$

$$S_B=2+4+6+8+\dots\dots\dots+2n \qquad =n^2+n$$

$$S_A-S_B=1+3+5+\dots\dots\dots+(2n-1) \qquad =n^2$$

Question 3

(a) Scale 10D (0, 3, 7, 8, 10)

Low Partial Credit:
- One correct step in induction
- Statement $P(1)$ true

Mid Partial Credit:
- Uses $(k+1)$ term on LHS
- Some work with $(k+1)$ term

High Partial Credit:
- Correct RHS
- No conclusion

(b) Scale 10D (0, 3, 7, 8, 10)

Low Partial Credit:
- Recognising 2 as common factor
- $\frac{n}{2}(n+1)$

Mid Partial Credit:
- Correct equation from (a)
- Taking 2 out of series

High Partial Credit:
- Work not completed

OR

(b) Scale 10D (0, 3, 7, 8, 10) [when series tested as an AP]

Low Partial Credit:
- Recognition of $a = 2$
- Recognition of $d = 2$
- Correct AP formula only

Mid Partial Credit:
- Some substitution into correct formula

High Partial Credit:
- Work not fully simplified
- Answer not in required form
- S_n missing

(c) Scale 5B (0, 3, 5)

Partial Credit:
- $S_B - S_A$ indicated
- $S_B + S_A$
- Use of correct series from (b)

Note: Must use result from (a) and (b) here