Question 6 **(25 marks)**

The n^{th} term of a sequence is $T_n = \ln a^n$, where $a > 0$ and a is a constant.

(a) **(i)** Show that $T_1,\ T_2,$ and T_3 are in arithmetic sequence.

(ii) Prove that the sequence is arithmetic and find the common difference.

(b) Find the value of a for which $T_1 + T_2 + T_3 + \cdots + T_{98} + T_{99} + T_{100} = 10\,100$.

(c) Verify that, for all values of a,

$$\left(T_1 + T_2 + T_3 + \cdots + T_{10}\right) + 100d = \left(T_{11} + T_{12} + T_{13} + \cdots + T_{20}\right),$$

where d is the common difference of the sequence.

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(a) **(i)** Show that T_1, T_2, and T_3 are in arithmetic sequence.

$T_1 = \ln a, \quad T_2 = \ln a^2 = 2\ln a, \quad T_3 = \ln a^3 = 3\ln a.$

$T_2 - T_1 = 2\ln a - \ln a = \ln a$

$T_3 - T_2 = 3\ln a - 2\ln a = \ln a$

$T_3 - T_2 = T_2 - T_1$. Hence, terms are in arithmetic sequence.

(ii) Prove that the sequence is arithmetic and find the common difference.

$T_n = \ln a^n = n\ln a,$

$T_{n-1} = \ln a^{n-1} = (n-1)\ln a.$

$T_n - T_{n-1} = n\ln a - (n-1)\ln a = \ln a$, (a constant).

Hence, the sequence is arithmetic.

Common difference: $T_n - T_{n-1} = \ln a$

(b) Find the value of a for which $T_1 + T_2 + T_3 + \cdots + T_{98} + T_{99} + T_{100} = 10\,100$.

$$T_1 + T_2 + T_3 + \cdots + T_{98} + T_{99} + T_{100} = 10\,100$$

$$\Rightarrow \ln a + 2\ln a + 3\ln a + \cdots + 100\ln a = 10\,100$$

$$\Rightarrow \frac{100}{2}\left[2\ln a + (100-1)\ln a\right] = 10\,100$$

$$\Rightarrow 50\left[101\ln a\right] = 10\,100$$

$$\Rightarrow 5050\ln a = 10\,100$$

$$\Rightarrow \ln a = 2$$

$$\Rightarrow a = e^2 = 7\cdot389$$

(c) Verify that, for all values of a,

$$\left(T_1 + T_2 + T_3 + \cdots + T_{10}\right) + 100d = \left(T_{11} + T_{12} + T_{13} + \cdots + T_{20}\right),$$

where d is the common difference of the sequence.

$$\begin{aligned}\left(T_1 + T_2 + T_3 + \cdots + T_{10}\right) + 100d &= (T_1 + 10d) + (T_2 + 10d) + (T_3 + 10d) + \cdots + (T_{10} + 10d) \\ &= T_{11} + T_{12} + T_{13} + \cdots + T_{20}.\end{aligned}$$

OR

$$\begin{aligned}\left(T_1 + T_2 + T_3 + \cdots + T_{10}\right) + 100d &= (\ln a + 2\ln a + 3\ln a + \cdots + 10\ln a) + 100\ln a \\ &= \tfrac{10}{2}\left(2\ln a + (10-1)\ln a\right) + 100\ln a \\ &= 5\left(11\ln a\right) + 100\ln a \\ &= 155\ln a\end{aligned}$$

$$\begin{aligned}\left(T_{11} + T_{12} + T_{13} + \cdots + T_{20}\right) &= 11\ln a + 12\ln a + 13\ln a + \cdots + 20\ln a \\ &= \tfrac{10}{2}\left(22\ln a + (10-1)\ln a\right) \\ &= 5\left(31\ln a\right) \\ &= 155\ln a\end{aligned}$$

Hence, L.H.S = R.H.S

Question 6

NOTE: When particular values are used in ALL sections give Low Partial Credit at most each time

(a)(i) Scale 10C (0, 5, 7, 10)

Low Partial Credit:

- Only one term correct

High Partial Credit:

- Either $(T_2 - T_1)$ or $(T_3 - T_2)$ correct

(a)(ii) Scale 5C (0, 3, 4, 5)

Low Partial Credit:

- Uses two consecutive general terms
- Recognition of common difference and no more

High Partial Credit:

- Shows series arithmetic but does not state common difference

(b) Scale 5C (0, 3, 4, 5)

Low Partial Credit:

- Writes three or more terms in form of n and $\ln a$
- Correct AP formula stated
- Correct T_n formula

High Partial Credit:

- Correct substitution into formula
- $\ln a = 2$ and does not finish

Note: accept $a = e^2$ for full marks

(c) Scale 5C (0, 3, 4, 5)

Low Partial Credit:

- Recognising $T_{11} = T_1 + 10d$ or similar work

High Partial Credit:

- LHS correct in terms of $\ln a$
- RHS correct in terms of $\ln a$

Note: log is not needed in first solution box