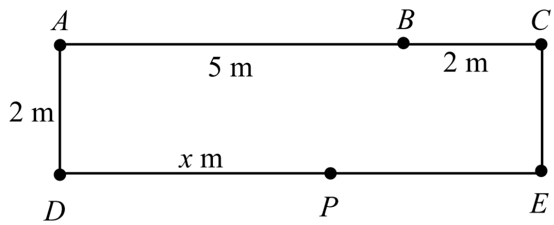
Question 7 **(40 marks)**

(a) Three natural numbers a, b and c, such that $a^2 + b^2 = c^2$, are called a Pythagorean triple.

(i) Let $a = 2n+1$, $b = 2n^2 + 2n$ and $c = 2n^2 + 2n + 1$.
Pick one natural number n and verify that the corresponding values of a, b and c form a Pythagorean triple.

(ii) Prove that $a = 2n+1$, $b = 2n^2 + 2n$ and $c = 2n^2 + 2n + 1$, where $n \in \mathbb{N}$, will always form a Pythagorean triple.

(b) $ADEC$ is a rectangle with $|AC| = 7$ m and $|AD| = 2$ m, as shown.
B is a point on $[AC]$ such that $|AB| = 5$ m.
P is a point on $[DE]$ such that $|DP| = x$ m.



(i) Let $f(x) = |PA|^2 + |PB|^2 + |PC|^2$.
Show that $f(x) = 3x^2 - 24x + 86$, for $0 \le x \le 7$, $x \in \mathbb{R}$.

(ii) The function $f(x)$ has a minimum value at $x = k$.
Find the value of k and the minimum value of $f(x)$.

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(i) Let $a = 2n+1$, $b = 2n^2 + 2n$ and $c = 2n^2 + 2n + 1$.
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Let $n = 1$:

$a = 2n+1 \Rightarrow a = 2(1)+1 = 3$

$b = 2n^2 + 2n \Rightarrow b = 2(1)^2 + 2(1) = 4$

$c = 2n^2 + 2n + 1 \Rightarrow c = 2(1)^2 + 2(1) + 1 = 5$

$3^2 + 4^2 = 5^2 \Rightarrow a^2 + b^2 = c^2$

(ii) Prove that $a = 2n+1$, $b = 2n^2 + 2n$ and $c = 2n^2 + 2n + 1$, where $n \in \mathbb{N}$, will always form a Pythagorean triple.

$a^2 = (2n+1)^2 = 4n^2 + 4n + 1$

$b^2 = (2n^2 + 2n)^2 = 4n^4 + 8n^3 + 4n^2$

$a^2 + b^2 = 4n^4 + 8n^3 + 8n^2 + 4n + 1$

$c^2 = (2n^2 + 2n + 1)^2$

$\quad = 4n^4 + 8n^3 + 8n^2 + 4n + 1$

$\quad = a^2 + b^2$

(b) *ADEC* is a rectangle with $|AC| = 7$ m and $|AD| = 2$ m, as shown.
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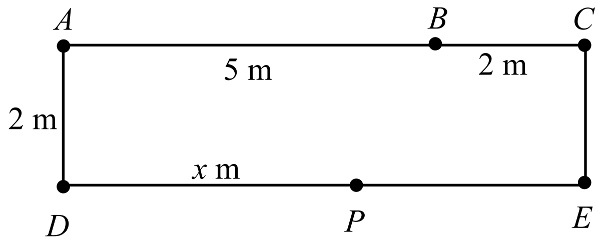

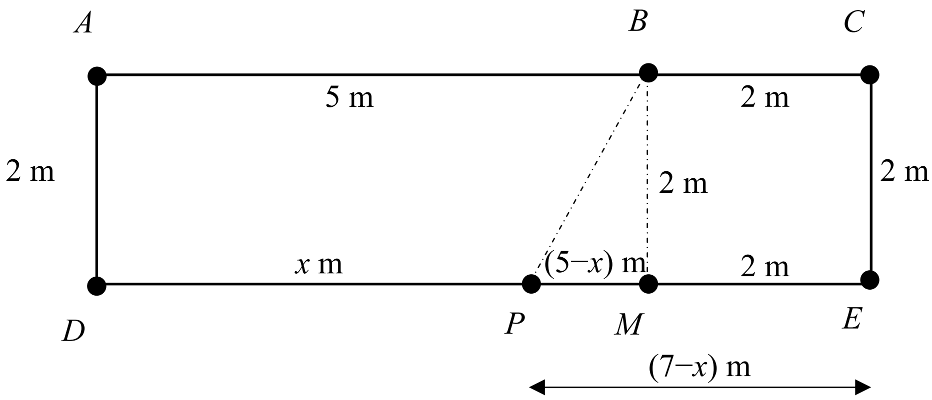

(i) Let $f(x) = |PA|^2 + |PB|^2 + |PC|^2$.
Show that $f(x) = 3x^2 - 24x + 86$, for $0 \le x \le 7$, $x \in \mathbb{R}$.

$$|PM| = |PE| - |ME|$$
$$= (7 - x) - 2$$
$$= (5 - x)$$

$$f(x) = |PA|^2 + |PB|^2 + |PC|^2$$
$$= \left[|PD|^2 + |DA|^2\right] + \left[|PM|^2 + |MB|^2\right] + \left[|PE|^2 + |EC|^2\right]$$
$$= x^2 + 2^2 + \left((5 - x)^2 + 2^2\right) + \left((7 - x)^2 + 2^2\right)$$
$$= x^2 + 4 + 25 - 10x + x^2 + 4 + 49 - 14x + x^2 + 4$$
$$= 3x^2 - 24x + 86$$

(ii) The function $f(x)$ has a minimum value at $x = k$.
Find the value of k and the minimum value of $f(x)$.

$$f(x) = 3x^2 - 24x + 86$$
$$f'(x) = 6x - 24$$
$$f''(x) = 6 > 0 \quad \Rightarrow \text{ minimum}$$
$$f'(x) = 0 \Rightarrow 6x - 24 = 0 \Rightarrow x = 4 = k$$
$$f(4) = 3(4)^2 - 24(4) + 86 = 38$$

OR

$$\begin{aligned} f(x) &= 3x^2 - 24x + 86 \\ &= 3\left(x^2 - 8x + \frac{86}{3}\right) \\ &= 3\left[\left(x^2 - 8x + 16\right) + \frac{38}{3}\right] \\ &= 3\left[\left(x-4\right)^2 + \frac{38}{3}\right] \end{aligned}$$

At $x = 4 \Rightarrow$ minimum value for $f(x)$

$$\begin{aligned} f(4) &= 3x^2 - 24x + 86 \\ &= 3(4)^2 - 24(4) + 86 \\ &= 48 - 96 + 86 \\ &= 38 \end{aligned}$$

Question 7

(a)(i) Scale 10B (0, 5, 10)

Partial Credit:
- Correct substitution of chosen value
- Not squaring values

Note: Allow 10 marks for $n = 0$ and correct work in (a)(i)

(a)(ii) Scale 10D (0, 3, 7, 8, 10)

Low Partial Credit:
- a^2 or b^2 or c^2 expressed in terms of n

Mid Partial Credit:
- Any two terms

High Partial Credit:
- Three terms fully squared
- $(a^2 + b^2)$ fully worked out in terms of n

Notes for (a)(i) and (a)(ii):
- Mark particular case with scheme for (a)(i) wherever it occurs
- Mark general case with scheme for (a)(ii) wherever it occurs

(b)(i) Scale 5D (0, 2, 3, 4, 5)

Low Partial Credit:
- Expression for either $|PA|^2$ or $|PB|^2$ or $|PC|^2$ in terms of x
- Any appropriate construction line, e.g. the line PM

Mid Partial Credit:
- Correct expression of two sides in terms of x

High Partial Credit:
- Correct expression of three sides in terms of x
- Correct expression of function in x not simplified

(b)(ii) Scale 15C (0, 7, 10, 15)

Low Partial Credit:
- Stating $f'(x) = 0$ with no work
- Any correct differentiation

High Partial Credit:
- Finding value of x

OR

(b)(ii) Scale 15C (0, 7, 10, 15)

Low Partial Credit:
- 3 as factor

High Partial Credit:
- Finding value of x