Question 9 **(60 marks)**

Ciarán is preparing food for his baby and must use cooled boiled water. The equation $y = Ae^{kt}$ describes how the boiled water cools. In this equation:

- t is the time, in minutes, from when the water boiled,
- y is the *difference* between the water temperature and room temperature at time t, measured in degrees Celsius,
- A and k are constants.

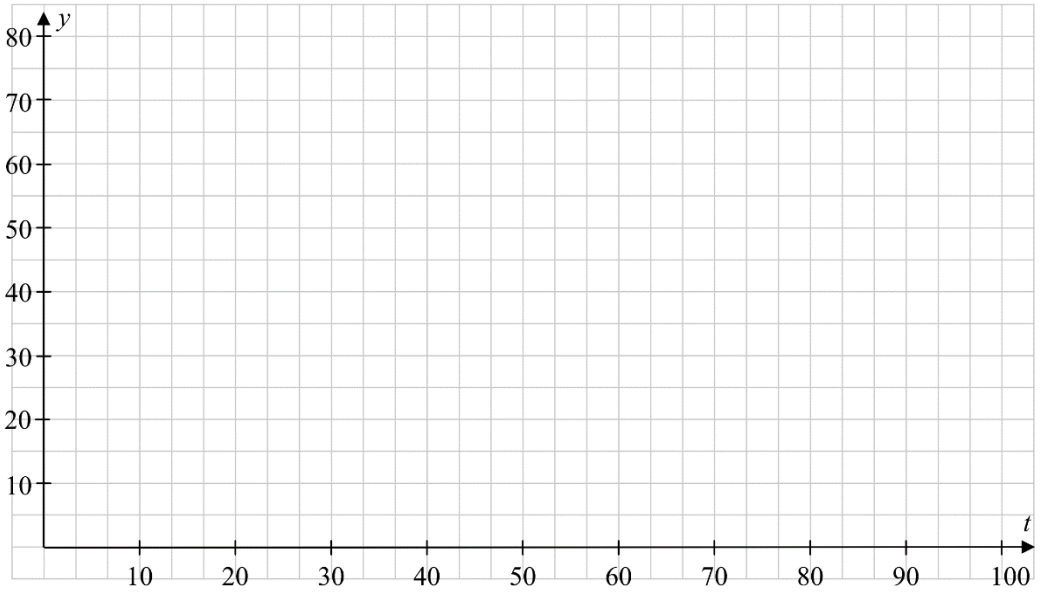
The temperature of the water when it boils is $100°C$ and the room temperature is a constant $23°C$.

(a) Write down the value of the temperature difference, y, when the water boils, and find the value of A.

(b) After five minutes, the temperature of the water is $88°C$.
Find the value of k, correct to three significant figures.

(c) Ciarán prepares the food for his baby when the water has cooled to $50°C$. How long does it take, correct to the nearest minute, for the water to cool to this temperature?

(d)



(e) **(i)** On the same diagram, sketch a curve $g(t) = Ae^{mt}$, showing the water cooling at a *faster* rate, where A is the value from part **(a)**, and m is a constant. Label each graph clearly.

(ii) Suggest one possible value for m for the sketch you have drawn and give a reason for your choice.

(f) **(i)** Find the rates of change of the function $f(t)$ after 1 minute and after 10 minutes.
Give your answers correct to two decimal places.

(ii) Show that the rate of change of $f(t)$ will always increase over time.

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The temperature of the water when it boils is $100°C$ and the room temperature is a constant $23°C$.

(a) Write down the value of the temperature difference, y, when the water boils, and find the value of A.

$y = 100 - 23 = 77$ at $t = 0$

$y = Ae^{kt} \Rightarrow 77 = Ae^0 \Rightarrow A = 77$

(b) After five minutes, the temperature of the water is $88°C$.
Find the value of k, correct to three significant figures.

At $t = 5$, $y = 88 - 23 = 65$

$y = 77e^{kt} \Rightarrow 65 = 77e^{5k} \Rightarrow 5k = \ln\frac{65}{77} = -0{\cdot}169418$

$\Rightarrow k = -0{\cdot}03388 \approx -0{\cdot}0339$

(c) Ciarán prepares the food for his baby when the water has cooled to $50°C$. How long does it take, correct to the nearest minute, for the water to cool to this temperature?

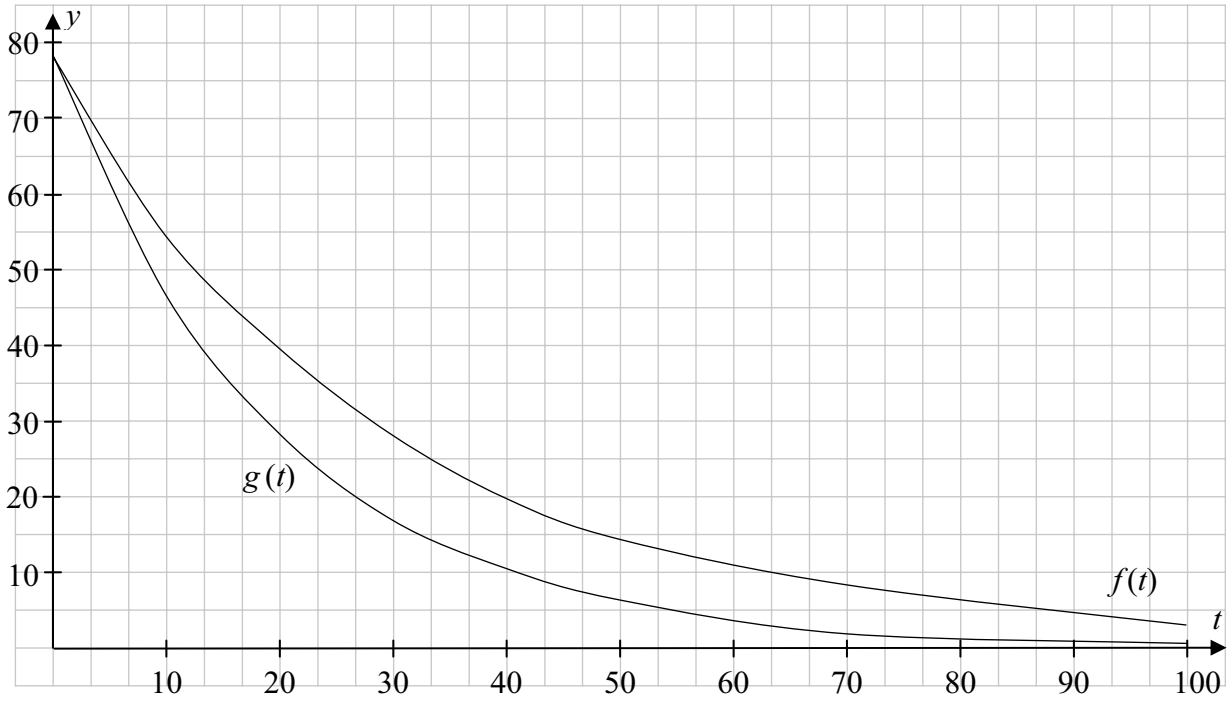
$y = 50 - 23 = 27$

$27 = 77e^{-0{\cdot}0339t} \Rightarrow 0{\cdot}0339t = \ln\frac{77}{27} = 1{\cdot}047969$

$\Rightarrow t = 30{\cdot}9 \approx 31$ minutes

(d) Using your values for A and k, sketch the curve $f(t) = Ae^{kt}$ for $0 \leq t \leq 100$, $t \in \mathbb{R}$.

$t = 0 \Rightarrow y = 77e^{-0\cdot0339(0)} = 77$; (0, 77)
$t = 10 \Rightarrow y = 77e^{-0\cdot0339(10)} = 54\cdot9$; (10, 55)
$t = 20 \Rightarrow y = 77e^{-0\cdot0339(20)} = 39\cdot1$; (20, 39)
$t = 30 \Rightarrow y = 77e^{-0\cdot0339(30)} = 27\cdot9$; (30, 28)
$t = 40 \Rightarrow y = 77e^{-0\cdot0339(40)} = 19\cdot8$; (40, 20)
$t = 50 \Rightarrow y = 77e^{-0\cdot0339(50)} = 14\cdot1$; (50, 14)
$t = 60 \Rightarrow y = 77e^{-0\cdot0339(60)} = 10\cdot1$; (60, 10)
$t = 70 \Rightarrow y = 77e^{-0\cdot0339(70)} = 7\cdot2$; (70, 7)
$t = 80 \Rightarrow y = 77e^{-0\cdot0339(80)} = 5\cdot1$; (80, 5)
$t = 90 \Rightarrow y = 77e^{-0\cdot0339(90)} = 3\cdot6$; (90, 4)
$t = 100 \Rightarrow y = 77e^{-0\cdot0339(100)} = 2\cdot6$; (100, 3)



(e) **(i)** On the same diagram, sketch a curve $g(t) = Ae^{mt}$, showing the water cooling at a *faster* rate, where A is the value from part **(a)**, and m is a constant. Label each graph clearly.

(ii) Suggest one possible value for m for the sketch you have drawn and give a reason for your choice.

Test $m = -0 \cdot 02$, $m = k = -0.0339$ and $m = -0 \cdot 05$

$m = -0 \cdot 02$, $t = 10 \Rightarrow y = 77e^{-0 \cdot 02(10)} = 63 \cdot 0$
$m = k = -0 \cdot 0339 \Rightarrow y = 54 \cdot 9$ (from table)
$m = -0 \cdot 05$, $t = 10 \Rightarrow y = 77e^{-0 \cdot 05(10)} = 46 \cdot 7$

Any value of $m < k$ for faster decay.

(f) **(i)** Find the rates of change of the function $f(t)$ after 1 minute and after 10 minutes. Give your answers correct to two decimal places.

$$y = 77e^{-0 \cdot 0339t} \Rightarrow \frac{dy}{dt} = -2 \cdot 6103e^{-0 \cdot 0339t}$$

$$t = 1, \ \frac{dy}{dt} = -2 \cdot 6103e^{-0 \cdot 0339} = -2 \cdot 52$$

$$t = 10, \ \frac{dy}{dt} = -2 \cdot 6103e^{-0 \cdot 339} = -1 \cdot 86$$

(ii) Show that the rate of change of $f(t)$ will always increase over time.

$$\frac{d^2y}{dt^2} = 0 \cdot 088e^{-0 \cdot 0339t} > 0 \Rightarrow \frac{dy}{dt} \text{ is increasing}$$

[25]

Question 9

(a) Scale 10C (0, 5, 7, 10)
Low Partial Credit:
- Value of y only
- Some use of 100 and/or 23

High Partial Credit:
- Correct substitution into equation
- A calculated from incorrect y

(b) Scale 10C (0, 5, 7, 10)
Low Partial Credit:
- Value of y only
- Some use of 88 and/or 23

High Partial Credit:
- Correct expression for k
- k calculated from incorrect y

(c) Scale 10C (0, 5, 7, 10)
Low Partial Credit:
- Value of y only

High Partial Credit:
- Correct expression for t
- t calculated from incorrect y

(d) Scale 15C (0, 7, 10, 15)
Low Partial Credit:
- Any one point identified
- Graph of correct shape, even if no point correct or no point calculated
- Accept candidates value of k

Note: all graphs may not be the same, due to different values of A and k

High Partial Credit:
- Three points correctly plotted, but graph incomplete or no graph

Note: do not accept straight line graph

(e)(i) and (e)(ii)

Scale 5C (0, 3, 4, 5)

Low Partial Credit:

- Any attempt at similar graph
- No graph but correct deduction

High Partial Credit:

- Correct graph plotted but graph incomplete, or no graph

(f)(i) Scale 5C (0, 3, 4, 5)

Low Partial Credit:

- Indication of differentiation i.e. $\frac{dy}{dt}$, $\frac{dx}{dt}$ or $f'(t)$ (i.e. differentiation with respect to t)
- Treats e as x in differentiation

High Partial Credit:

- One value of $\frac{dy}{dt}$ indicated

(f)(ii) Scale 5C (0, 3, 4, 5)

Low Partial Credit:

- Attempt at 2nd derivative
- Attempt at deduction from numerical values

High Partial Credit:

- Shows 2nd derivative positive