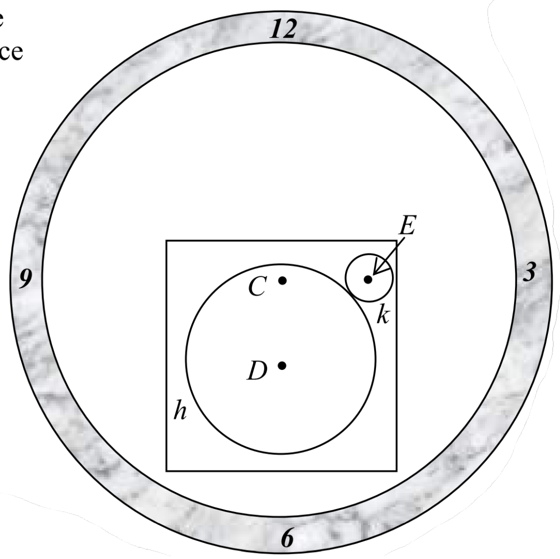
Question 9 **(60 marks)**

(a) The diagram shows a circular clock face, with the hands not shown. The square part of the clock face is glass so that the mechanism is visible. Two circular cogs, h and k, which touch externally are shown.



The point C is the centre of the clock face. The point D is the centre of the larger cog, h, and the point E is the centre of the smaller cog, k.

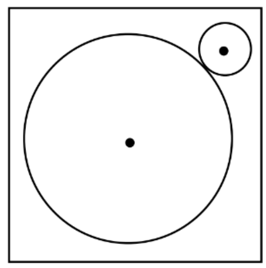
(i) In suitable co-ordinates, the equation of the circle h is
$$x^2 + y^2 + 4x + 6y - 19 = 0.$$
Find the radius of h, and the co-ordinates of its centre, D.

(ii) The point E has co-ordinates $(3, 2)$. Find the radius of the circle k.

(iii) Show that the distance from $C(-2, 2)$ to the line DE is half the length of $[DE]$.

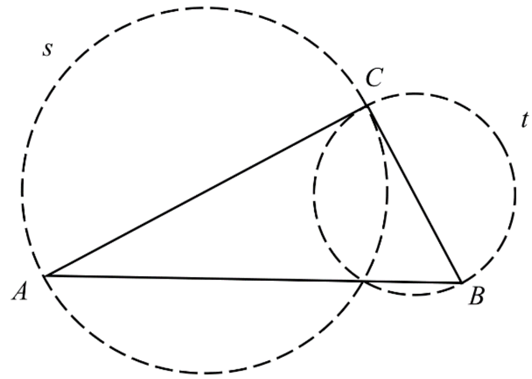
(iv) The translation which maps the midpoint of $[DE]$ to the point C maps the circle k to the circle j. Find the equation of the circle j.

(v) The glass square is of side length l. Find the smallest whole number l such that the two cogs, h and k, are fully visible through the glass.



(b) The triangle ABC is right-angled at C.

The circle s has diameter $[AC]$ and the circle t has diameter $[CB]$.



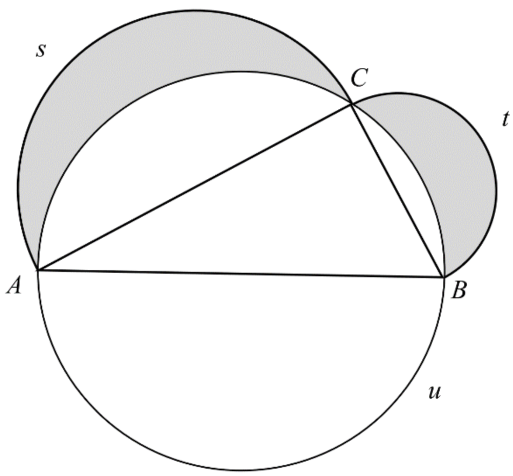
(i) Draw the circle u which has diameter $[AB]$.

(ii) Prove that in any right-angles triangle ABC, areas of the circles s and t.

(iii) The diagram shows the right-angled triangle ABC and arcs of the circles s, t and u.

Each of the shaded areas in the diagram is called a lune, a crescent-shaped area bounded by arcs of the circles.

Prove that the sum of the areas of the two shaded lunes is equal to the area of the triangle ABC.



Question 9 **(60 marks)**

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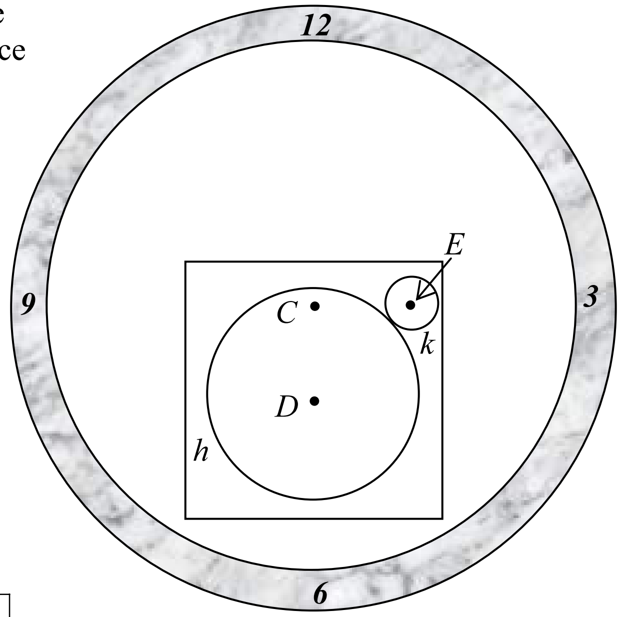


The point C is the centre of the clock face. The point D is the centre of the larger cog, h, and the point E is the centre of the smaller cog, k.

(i) In suitable co-ordinates the equation of the circle h is

$$x^2 + y^2 + 4x + 6y - 19 = 0.$$

Find the radius of h and the co-ordinates of its centre, D.

$r_1 = \sqrt{4+9+19} = \sqrt{32} = 4\sqrt{2}$

$D(-2, -3)$

(ii) The point E has co-ordinates $(3, 2)$. Find the radius of the circle k.

$|DE| = \sqrt{(3+2)^2 + (2+3)^2} = \sqrt{50} = 5\sqrt{2}$

$r_1 + r_2 = |DE| \Rightarrow 4\sqrt{2} + r_2 = 5\sqrt{2} \Rightarrow r_2 = \sqrt{2}$

(iii) Show that the distance from $C(-2, 2)$ to the line DE is half the length of $[DE]$.

Slope $DE = \dfrac{2+3}{3+2} = 1$

Equation DE : $y + 3 = 1(x+2) \Rightarrow x - y - 1 = 0$

Distance from C to DE: $p = \left|\dfrac{-2-2-1}{\sqrt{1+1}}\right| = \left|\dfrac{5}{\sqrt{2}}\right| = \dfrac{5\sqrt{2}}{2} = \dfrac{1}{2}|DE|$

(iv) The translation which maps the midpoint of $[DE]$ to the point C maps the circle k to the circle j. Find the equation of the circle j.

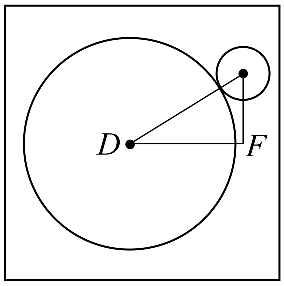
$$\text{Midpoint } [DE] = \left(\frac{-2+3}{2}, \frac{-3+2}{2}\right) = \left(\frac{1}{2}, -\frac{1}{2}\right)$$

$$\left(\tfrac{1}{2}, -\tfrac{1}{2}\right) \to (-2, 2) \text{ maps } (3,\ 2) \to \left(\tfrac{1}{2}, \tfrac{9}{2}\right)$$

$$j: \left(x - \tfrac{1}{2}\right)^2 + \left(y - \tfrac{9}{2}\right)^2 = \left(\sqrt{2}\right)^2 = 2$$

$$4x^2 + 4y^2 - 4x - 36y + 74 = 0$$

(v) The glass square is of side length l. Find the smallest whole number l such that the two cogs, h and k, are fully visible through the glass.



$D(-2, -3), \quad F(3, -3)$

$|DF| = 5$

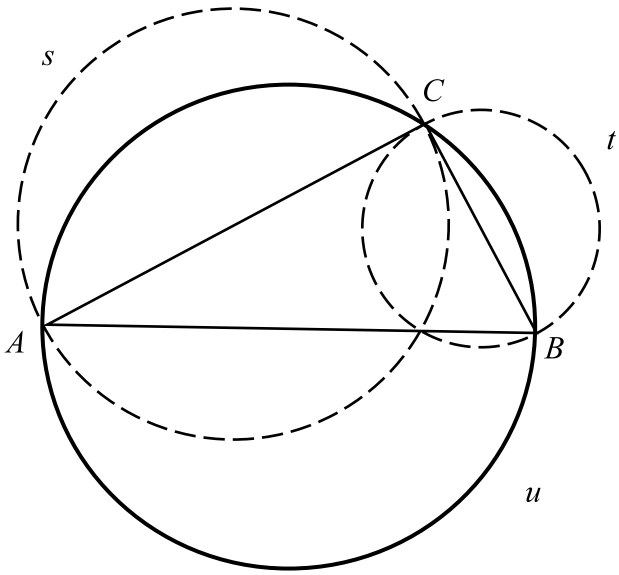
Length: $r_1 + |DF| + r_2 = 4\sqrt{2} + 5 + \sqrt{2} = 5\sqrt{2} + 5 = 12 \cdot 07$

$l = 13$

(b) The triangle ABC is right-angled at C.

The circle s has diameter $[AC]$ and the circle t has diameter $[CB]$.

(i) Draw the circle u which has diameter $[AB]$.



(ii) Prove that in any right-angles triangle *ABC*, the area of the circle *u* equals the sum of the areas of the circles *s* and *t*.

Triangle *ABC* is right-angled:

$$|AB|^2 = |AC|^2 + |CB|^2$$

$$\Rightarrow \frac{\pi}{4}\left(|AB|^2\right) = \frac{\pi}{4}\left(|AC|^2 + |CB|^2\right)$$

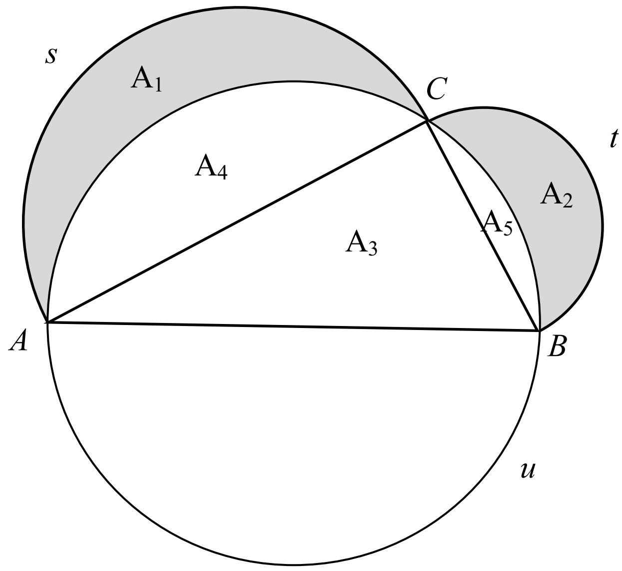
$$\Rightarrow \pi\left(\frac{|AB|}{2}\right)^2 = \pi\left(\frac{|AC|}{2}\right)^2 + \pi\left(\frac{|CB|}{2}\right)^2$$

Thus, area of *u* = area of *s* + area of *t*.

(iii) The diagram shows the right-angled triangle *ABC* and arcs of the circles *s*, *t* and *u*.

Each of the shaded areas in the diagram is called a lune, a crescent-shaped area bounded by arcs of the circles.

Prove that the sum of the areas of the two shaded lunes is equal to the area of the triangle *ABC*.



$$\frac{1}{2} \text{ area of } u = \frac{1}{2}(\text{ area of } s + \text{area of } t)$$

$$\Rightarrow \mathrm{A}_3 + \mathrm{A}_4 + \mathrm{A}_5 = (\mathrm{A}_1 + \mathrm{A}_4) + (\mathrm{A}_2 + \mathrm{A}_5)$$

$$\Rightarrow \mathrm{A}_3 = \mathrm{A}_1 + \mathrm{A}_2$$

Question 9 **(60 marks)**

(a)(i) Scale 15C (0, 5, 10, 15)

Low Partial Credit:

- Effort at relating one or more coefficients of given equation to general equation of circle
- Effort at completing square(s)

High Partial Credit:

- Either radius or centre correct
- Substantive work with one critical error

(ii) Scale 10C (0, 3, 7, 10)

Low Partial Credit:

- Effort at finding $|DE|$
- Length of line segment formula
- Indicates some understanding of $r_1 + r_2 = |DE|$

High Partial Credit:

- $r_1 + r_2 = |DE|$ or equivalent with known values substituted

(iii) Scale 10C (0, 3, 7, 10)

Low Partial Credit:

- Slope DE
- Equation DE and stops
- Formula for slope and /or equation of DE
- Perpendicular distance formula

High Partial Credit:

- Values inserted into perpendicular distance formula
- No conclusion stated or implied

(iv) Scale 5C (0, 2, 3, 5)

Low Partial Credit:

- Effort to find midpoint of DE
- Centre found from scaled drawing

High Partial Credit:

- Centre of j found and inserted into equation of circle i.e radius omitted

(v) Scale 5C (0, 2, 3, 5)

Low Partial Credit:

- Effort to find F
- Indication length $r_1 + r_2 + |DF|$ (or equivalent)

High Partial Credit:

- F found

(b)(i) Scale 5B (0, 2, 5)

Partial Credit:

- Circle containing A and B but lacking in accuracy

(ii) Scale 5C (0, 2, 3, 5)

Low Partial Credit:

- Pythagoras stated or implied
- Effort at finding area of s or t or u

High Partial Credit

- Correct expression for area of any circle

  e.g area $u = \frac{\pi}{4}\left(|AB|^2\right)$ or $\pi\left(\frac{|AB|}{2}\right)^2$

(iii) Scale 5C (0, 2, 3, 5)

Low Partial Credit:

- Statement using result from (b)(ii)
- Recognising half the area of s or half the area t can be expressed in terms of two component areas
- Recognising half area of u can be expressed in terms of three components

High Partial Credit

- Correct expression for two of the relevant areas