Question 8 **(65 marks)**

In basketball, players often have to take free throws. When Michael takes his first free throw in any game, the probability that he is successful is 0·7.
For all subsequent free throws in the game, the probability that he is successful is:

- 0·8 if he has been successful on the previous throw
- 0·6 if he has been unsuccessful on the previous throw.

(a) Find the probability that Michael is successful (S) with all three of his first three free throws in a game.

(b) Find the probability that Michael is unsuccessful (U) with his first two free throws and successful with the third.

(c) List all the ways that Michael could be successful with his third free throw in a game and hence find the probability that Michael is successful with his third free throw.

(d) **(i)** Let p_n be the probability that Michael is successful with his n^{th} free throw in the game (and hence $(1-p_n)$ is the probability that Michael is unsuccessful with his n^{th} free throw). Show that $p_{n+1} = 0 \cdot 6 + 0 \cdot 2 p_n$.

(ii) Assume that p is Michael's success rate in the long run; that is, for large values of n, we have $p_{n+1} \approx p_n \approx p$.
Using the result from part **(d) (i)** above, or otherwise, show that $p = 0 \cdot 75$.

(e) For all positive integers n, let $a_n = p - p_n$, where $p = 0 \cdot 75$ as above.

(i) Use the ratio $\dfrac{a_{n+1}}{a_n}$ to show that a_n is a geometric sequence with common ratio $\dfrac{1}{5}$.

(ii) Find the smallest value of n for which $p - p_n < 0 \cdot 00001$.

(f) You arrive at a game in which Michael is playing. You know that he has already taken many free throws, but you do not know what pattern of success he has had.

(i) Based on this knowledge, what is your estimate of the probability that Michael will be successful with his next free throw in the game?

(ii) Why would it **not** be appropriate to consider Michael's subsequent free throws in the game as a sequence of Bernoulli trials?

Question 8 **(65 marks)**

In basketball, players often have to take free throws. When Michael takes his first free throw in any game, the probability that he is successful is 0·7.
For all subsequent free throws in the game, the probability that he is successful is:

- 0·8 if he has been successful on the previous throw
- 0·6 if he has been unsuccessful on the previous throw.

(a) Find the probability that Michael is successful (S) with all three of his first three free throws in a game.

$$\text{P}(\text{S}, \text{S}, \text{S}) = 0 \cdot 7 \times 0 \cdot 8 \times 0 \cdot 8 = 0 \cdot 448$$

(b) Find the probability that Michael is unsuccessful (U) with his first two free throws and successful with the third.

$$\text{P}(\text{U}, \text{U}, \text{S}) = 0 \cdot 3 \times 0 \cdot 4 \times 0 \cdot 6 = 0 \cdot 072$$

(c) List all the ways that Michael could be successful with his third free throw in a game and hence find the probability that Michael is successful with his third free throw.

S, S, S   U, U, S   S, U, S   U, S, S

$$\text{P}(\text{S}, \text{S}, \text{S}) = 0 \cdot 7 \times 0 \cdot 8 \times 0 \cdot 8 = 0 \cdot 448$$
$$\text{P}(\text{U}, \text{U}, \text{S}) = 0 \cdot 3 \times 0 \cdot 4 \times 0 \cdot 6 = 0 \cdot 072$$
$$\text{P}(\text{S}, \text{U}, \text{S}) = 0 \cdot 7 \times 0 \cdot 2 \times 0 \cdot 6 = 0 \cdot 084$$
$$\text{P}(\text{U}, \text{S}, \text{S}) = 0 \cdot 3 \times 0 \cdot 6 \times 0 \cdot 8 = 0 \cdot 144$$

$$\text{P} = 0 \cdot 448 + 0 \cdot 072 + 0 \cdot 084 + 0 \cdot 144 = 0 \cdot 748$$

(d) **(i)** Let p_n be the probability that Michael is successful with his n^{th} free throw in the game (and hence $(1-p_n)$ is the probability that Michael is unsuccessful with his n^{th} free throw). Show that $p_{n+1} = 0\cdot 6 + 0\cdot 2p_n$.

$$\begin{aligned} p_{n+1} &= \mathrm{P}(\mathrm{S,S}) + \mathrm{P}(\mathrm{U,S}) \\ &= p_n \times 0\cdot 8 + (1-p_n)0\cdot 6 \\ &= 0\cdot 6 + 0\cdot 2p_n \end{aligned}$$

(ii) Assume that p is Michael's success rate in the long run; that is, for large values of n, we have $p_{n+1} \approx p_n \approx p$.
Using the result from part **(d) (i)** above, or otherwise, show that $p = 0\cdot 75$.

$$\begin{aligned} & p \approx p_n \approx p_{n+1} = 0\cdot 6 + 0\cdot 2p_n \\ & \Rightarrow 0\cdot 8p_n = 0\cdot 6 \\ & \Rightarrow p_n = \frac{0\cdot 6}{0\cdot 8} = 0\cdot 75 = p \end{aligned}$$

(e) For all positive integers n, let $a_n = p - p_n$, where $p = 0\cdot 75$ as above.

(i) Use the ratio $\dfrac{a_{n+1}}{a_n}$ to show that a_n is a geometric sequence with common ratio $\dfrac{1}{5}$.

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{p - p_{n+1}}{p - p_n} \\ &= \frac{0\cdot 75 - (0\cdot 6 + 0\cdot 2p_n)}{0\cdot 75 - p_n} \\ &= \frac{0\cdot 15 - 0\cdot 2p_n}{5(0\cdot 15 - 0\cdot 2p_n)} = \frac{1}{5} \end{aligned}$$

(ii) Find the smallest value of n for which $p - p_n < 0{\cdot}00001$.

$$a_n = p - p_n$$
$$a_1 = p - p_1 = 0\cdot 75 - 0\cdot 7 = 0\cdot 05$$

$$ar^{n-1} = 0\cdot 05(0\cdot 2)^{n-1} < 0\cdot 00001$$
$$(n-1)\ln 0\cdot 2 < \ln 0\cdot 0002$$
$$\Rightarrow n-1 > \frac{\ln 0\cdot 0002}{\ln 0\cdot 2} = 5\cdot 29$$
$$\Rightarrow n > 6\cdot 29$$

$$n = 7$$

(f) You arrive at a game in which Michael is playing. You know that he has already taken many free throws, but you do not know what pattern of success he has had.

(i) Based on this knowledge, what is your estimate of the probability that Michael will be successful with his next free throw in the game?

Answer: $0{\cdot}75$ **or** p

(ii) Why would it **not** be appropriate to consider Michael's subsequent free throws in the game as a sequence of Bernoulli trials?

Events not independent

Question 8 **(65 marks)**

(a) Scale 10C (0, 4, 8, 10)

Low *Partial Credit:*

- One correct probability

High Partial Credit:

- Identifies all three probabilities correctly
- Three probabilities multiplied of which two are correct

(b) Scale 10C (0, 4, 8, 10)

Low *Partial Credit:*

- One correct probability

High Partial Credit:

- Identifies all three probabilities correctly
- Three probabilities multiplied of which two are correct

(c) Scale 15D (0, 4, 7, 11, 15)

Low *Partial Credit:*

- Lists one new way

Mid Partial Credit:

- Full listing only
- One new probability

High Partial Credit:

- Sum of three probabilities
- Identifies all four probabilities correctly

(d) (i) Scale 5C (0, 2, 4, 5)

Low *Partial Credit:*

- Indicates P(S, S) and/or P(U, S) or equivalent

High Partial Credit:

- Substitution into equation for p_{n+1}

(d) (ii) Scale 10B (0, 5, 10)

Partial Credit:

- Partial substitution into equation

(e)(i) Scale 5C (0, 2, 4, 5)

Low *Partial Credit:*

- a_{n+1} in terms of p and p_{n+1}
- $\frac{a_{n+1}}{a_n}$ in terms of p, p_n, and p_{n+1}

High Partial Credit:

- $\frac{a_{n+1}}{a_n}$ substituted

(e)(ii) Scale 5C (0, 2, 4, 5)

Low *Partial Credit:*

- a_1 in numerical form

High Partial Credit:

- ar^{n-1} substituted
- a_7 evaluated without checking a_6

(f) Scale 5B (0, 2, 5)

Partial Credit:

- (i) correct only or (ii) correct only