Question 7 **(55 marks)**

Sometimes it is possible to predict the future population in a city using a function.
The population in Sapphire City, over time, can be predicted using the following function:

$$p(t) = Se^{0 \cdot 1t} \times 10^6.$$

The population in Avalon, over time, can be predicted using the following function:

$$q(t) = 3 \cdot 9e^{kt} \times 10^6.$$

In the functions above, t is time, in years; $t = 0$ is the beginning of 2010; and both S and k are constants.

(a) The population in Sapphire City at the beginning of 2010 is 1 100 000 people.
Find the value of S.

(b) Find the predicted population in Sapphire City at the beginning of 2015.

(c) Find the predicted change in the population in Sapphire City during 2015.

(d) The predicted population in Avalon at the beginning of 2011 is 3 709 795 people.
Write down and solve an equation in k to show that $k = -0 \cdot 05$, correct to 2 decimal places.

(e) Find the year during which the populations in both cities will be equal.

(f) Find the predicted average population in Avalon from the beginning of 2010 to the beginning of 2025.

(g) Use the function $q(t) = 3 \cdot 9e^{-0 \cdot 05t} \times 10^6$ to find the predicted rate of change of the population in Avalon at the beginning of 2018.

Q7	Model Solution – 55 Marks	Marking Notes
(a)	$Se^{\cdot 1(0)} \times 10^6 = 1100000$ $S = 1{\cdot}1$	**Scale 10B (0, 4, 10)** *Partial Credit* • equation in S with substitution
(b)	$p(5) = 1{\cdot}1e^{0\cdot 1(5)} \times 10^6$ $= 1{\cdot}813593 \times 10^6$ $= 1813593$	**Scale 10B (0, 4, 10)** *Partial Credit* • substitution into formula for $p(5)$
(c)	$p(6) = 1{\cdot}1e^{0\cdot 6} \times 10^6$ $p(5) = 1{\cdot}1e^{0\cdot 5} \times 10^6$ $p(6) - p(5) = (1{\cdot}1e^{0\cdot 6} - 1{\cdot}1e^{0\cdot 5}) \times 10^6$ $= 0{\cdot}1907372 \times 10^6$ $= 190737$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* • substitution into formula for $p(6)$ • use of $p(5)$ from previous part • $p(6) - p(5)$ written or implied *High partial Credit* • Formulates $p(6) - p(5)$ with some substitution

(d)	$$q(t) = 3{\cdot}9e^{kt} \times 10^6$$ $$3709795 = 3{\cdot}9e^{k} \times 10^6$$ $$\frac{3{\cdot}709795}{3{\cdot}9} = e^k$$ $$\log_e \frac{3{\cdot}709795}{3{\cdot}9} = k$$ $$k = -0{\cdot}0499 = -0{\cdot}05$$	**Scale 15C (0, 5, 10, 15)** *Low Partial Credit* • Either substitution into formula for k • Verifies k value only. *High Partial Credit* • relevant equation in k
(e)	$$p(t) = q(t)$$ $$1{\cdot}1e^{0{\cdot}1t} \times 10^6 = 3{\cdot}9e^{-0{\cdot}05t} \times 10^6$$ $$1{\cdot}1e^{0{\cdot}1t} = 3{\cdot}9e^{-0{\cdot}05t}$$ $$\frac{e^{0{\cdot}1t}}{e^{-0{\cdot}05t}} = \frac{3{\cdot}9}{1{\cdot}1}$$ $$e^{0{\cdot}15t} = \frac{39}{11}$$ $$\ln\frac{39}{11} = 0{\cdot}15t$$ $t = 8{\cdot}44$ years In 2018 both populations equal	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit* • $p(t) = q(t)$ written or implied *High Partial Credit* • relevant equation in t
(f)	$$\frac{1}{15}\int_0^{15} 3{\cdot}9e^{-0{\cdot}05t} \times 10^6 dt$$ $$\frac{1}{15}\left[\frac{3{\cdot}9}{-0{\cdot}05}e^{-0{\cdot}05(15)} - \frac{3{\cdot}9}{-0{\cdot}05}e^{-0{\cdot}05(0)}\right]$$ $$\times 10^6$$ $$2{\cdot}743694 \times 10^6$$ $$2743694$$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* • integral formulated (with limits) *High Partial Credit:* • integration with full substitution
(g)	$$q(t) = 3.9e^{-0.05t} \times 10^6$$ $$q'(t) = -0{\cdot}05(3{\cdot}9e^{-0{\cdot}05t} \times 10^6)$$ $$q'(8) = -0{\cdot}05\left(3{\cdot}9e^{-0{\cdot}05(8)} \times 10^6\right)$$ $$= -130712$$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit* • $q'(t)$ *High Partial Credit* • $q'(t)$ fully substituted