Question 4 **(25 marks)**

(a) Prove, using induction, that if n is a positive integer then

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta),\text{ where } i^2 = -1.$$

(b) Hence, or otherwise, find $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$ in its simplest form.

Q4	Model Solution – 25 Marks	Marking Notes
(a)	***P*(1)** $(\cos\theta + i\sin\theta)^1 = \cos(1\theta) + i\sin(1\theta)$ $\boldsymbol{P(k)}$: Assume $(\cos\theta + i\sin\theta)^k$ $= \cos(k\theta) + i\sin(k\theta)$ Test $\boldsymbol{P(k+1)}$: $(\cos\theta + i\sin\theta)^{k+1} =$ $= \cos(k+1)\theta + i\sin(k+1)\theta$ $(\cos\theta + i\sin\theta)^{k+1}$ $= (\cos\theta + i\sin\theta)^k.(\cos\theta + i\sin\theta)^1$ $= (\cos(k\theta) + i\sin(k\theta)).(\cos\theta + i\sin\theta)$ $= [\cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta]$ $+i[\cos(k\theta)\sin\theta + \cos\theta\sin(k\theta)]$ $= \cos(k+1)\theta + isin(k+1)\theta$ Thus the proposition is true for $n = k+1$ provided it is true for $n = k$ but it is true for $n = 1$ and therefore true for all positive integers.	**Scale 15D (0, 5, 7, 11, 15)** *Low Partial Credit:* Step $P(1)$ *Mid Partial Credit:* Step $P(k)$ or Step $P(k+1)$ *High Partial Credit:* Uses Step $P(k)$ to prove Step $P(k+1)$ Note: Accept Step $P(1)$, Step $P(k)$, Step $P(k+1)$ in any order *Full credit* –1: Omits conclusion but otherwise correct *Full credit*: $[r(\cos\theta + i\sin\theta)]^n$ $= r^n(\cos(n\theta) + i\sin(n\theta))$ proved correctly
(b)	$\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3 = 1\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^3$ $= \left(\cos(3)\frac{2\pi}{3} + i\sin(3)\frac{2\pi}{3}\right)$ $= (\cos 2\pi + i\sin 2\pi) =$ $1 + 0i$ $= 1$	**Scale 10C (0, 4, 8, 10)** *Low Partial Credit:* Modulus or argument correct Some correct multiplication Apply De Moivre correctly with incorrect modulus and argument *High Partial Credit:* $\left(\cos(3)\frac{2\pi}{3} + i\sin(3)\frac{2\pi}{3}\right)$ Multiplication correct but un-simplified *Full credit –1*: $\cos 2\pi + i\sin 2\pi$ or $\cos 360° + i\sin 360°$ **Accept**: Answer with reference to cube root of unity