Question 9 **(55 marks)**

The diagram below shows the first 4 steps of an infinite pattern which creates the *Sierpinski Triangle*. The sequence begins with a black equilateral triangle. Each step is formed by **removing** an equilateral triangle from the centre of each black triangle in the previous step, as shown. Each equilateral triangle that is removed is formed by joining the midpoints of the sides of a black triangle from the previous step.

(a) The table below shows the number of black triangles at each of the first 4 steps **and** the fraction of the original triangle remaining at each step. Complete the table.

Step	0	1	2	3
Number of black triangles	1			
Fraction of the original triangle remaining	1		$\frac{9}{16}$	

(b) **(i)** Write an expression in terms of n for the number of black triangles in step n of the pattern.

(ii) Step *k* is the first step of the pattern in which the number of black triangles **exceeds** one thousand million (i.e. 1×10^9) for the first time. Find the value of *k*.

(c) **(i)** Step *h* is the first step of the pattern in which the fraction of the original triangle remaining is less than $\frac{1}{100}$ of the original triangle. Find the value of *h*.

(ii) What fraction of the original triangle remains after an infinite number of steps of the pattern?

(d) **(i)** The side length of the triangle in Step 0 is 1 unit. The table below shows the total **perimeter** of all the black triangles in each of the first 5 steps.
Complete the table below.

Step	0	1	2	3	4
Perimeter	3		$\frac{27}{4}$		

(ii) Find the total perimeter of the black triangles in step 35 of the pattern.
Give your answer correct to the nearest unit.

(iii) Use your answers to part **(c)(ii)** and part **(d)(ii)** to comment on the total **area** and the total **perimeter** of the black triangles in step n of the pattern, as n tends to infinity.

Q9	Model Solution – 55 Marks	Marking Notes
(a)	<table><tr><th>Step</th><th>0</th><th>1</th><th>2</th><th>3</th></tr><tr><td>Triangles Remaining</td><td>1</td><td>3</td><td>9</td><td>27</td></tr><tr><td>Fraction of Original Triangle Remaining</td><td>1</td><td>$\frac{3}{4}$</td><td>$\frac{9}{16}$</td><td>$\frac{27}{64}$</td></tr></table>	
		Scale 10C (0, 4, 8, 10) *Low Partial Credit:* One correct entry *High Partial Credit:* Three correct entries *Full credit* –1: Answers as decimals
(b) (i)	3^n	**Scale 5B (0, 2, 5)** *Partial Credit:* $3n$ written n^3 written *Full credit* –1: 3^{n-1} written
(b) (ii)	$3^k > 1{,}000{,}000{,}000$ $\log_3 3^k > \log_3 1\,000\,000\,000$ $k\log_3 3 \quad > \log_3 1\,000\,000\,000$ $k > \log_3 1 \times 10^9$ $k > 18{\cdot}863$ $k = 19$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* $3^k > 1{,}000{,}000{,}000$ *High Partial Credit:* Inequality with k not written as an index **Note**: if $3k$ or k^3 from above used fully here then award low partial credit at most

(c) (i)	$\left(\frac{3}{4}\right)^h < \frac{1}{100}$ $\ln\left(\frac{3}{4}\right)^h < \ln\frac{1}{100}$ $h\ln\left(\frac{3}{4}\right) < \ln\frac{1}{100}$ $h > \frac{\ln\frac{1}{100}}{\ln\left(\frac{3}{4}\right)}$ $h > 16{\cdot}007$ $\Rightarrow h = 17$	**Scale 10C (0, 4, 8, 10)** *Low Partial Credit:* Correct answer without work $\left(\frac{3}{4}\right)^h$ or candidates ratio to the power of h $r = \frac{3}{4}$ Lists two or more terms *High Partial Credit:* Inequality with h not written as an index *Full credit* –1: $\left(\frac{3}{4}\right)^{h-1} < \frac{1}{100}$ and finishes correctly
(c) (ii)	$\lim_{n\to\infty}\left(\frac{3}{4}\right)^n = 0$ $\Longrightarrow$ Fraction remaining $= 0$	**Scale 5B (0, 2, 5)** *Partial Credit:* $\lim_{n\to\infty}$ Some use of $\frac{3}{4}$ *Full Credit*: Correct answer without work $\frac{1}{\infty}$ or equivalent

(d) (i)

Step	0	1	2	3	4
Perimeter	3	$\frac{9}{2}$	$\frac{27}{4}$	$\frac{81}{8}$	$\frac{243}{16}$

Scale 10C (0, 4, 8, 10)
Low Partial Credit:
One correct entry
All numerators correct with all incorrect denominators
All denominators correct with all incorrect numerators

High Partial Credit:
Two correct entries

(d) (ii)	Pattern: $\frac{3^1}{2^0}, \frac{3^2}{2^1}, \frac{3^3}{2^2} \dots\dots\dots . \frac{3^{n+1}}{2^n}$ $\therefore \text{step } 35 = \frac{3^{36}}{2^{35}}$ $= 4\ 368\ 329$ **Or** $T_{35} = \left(\frac{9}{2}\right)\left(\frac{3}{2}\right)^{34} = 4\ 368\ 329$ **Or** $T_{35} = (3)\left(\frac{3}{2}\right)^{35} = 4\ 368\ 329$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* Pattern identified Recognises $r = \frac{3}{2}$ Some relevant substitution into $T_n = ar^{n-1}$ $a = 3$ or $a = 4{\cdot}5$ *High Partial Credit:* Step 35 $= \frac{3^{36}}{2^{35}}$ or equivalent *Full credit* –1: $T_{35} = (3)\left(\frac{3}{2}\right)^{34}$
(d) (iii)	Area = 0 $\lim_{n\to\infty}\left(\frac{3^{n+1}}{2^n}\right) = \infty$ $\Longrightarrow \text{Perimeter } \to \infty$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* $\lim_{n\to\infty}\left(\frac{3^{n+1}}{2^n}\right)$ or equivalent Area is getting smaller Perimeter is increasing *High Partial Credit:* Area approaches 0 **Perimeter** $\to \infty$ identified Area is getting smaller and Perimeter is increasing