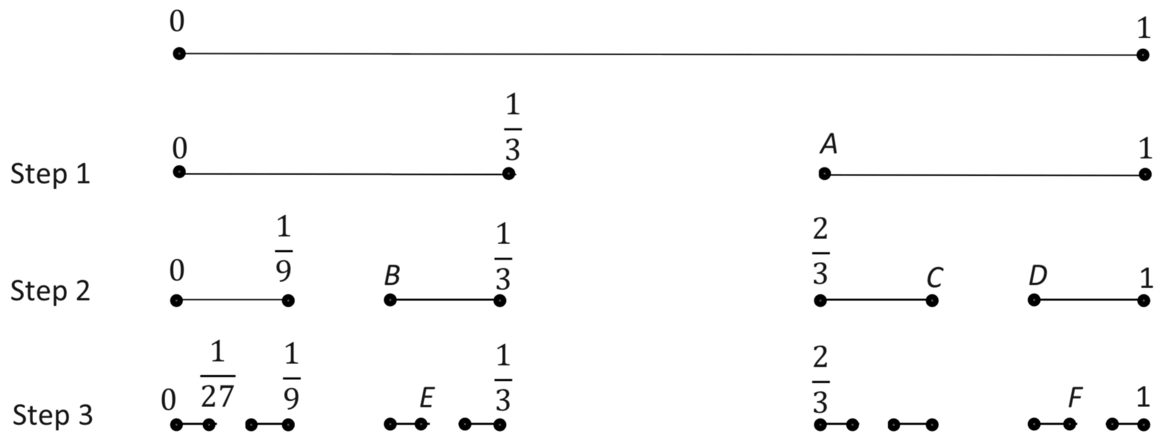
Question 7 **(45 marks)**

The closed line segment [0, 1] is shown below. The first three steps in the construction of the *Cantor Set* are also shown:

- Step 1 removes the open middle third of the line segment [0, 1] leaving two **closed line segments** (i.e. the end points of the segments remain in the *Cantor Set*)
- Step 2 removes the middle third of the two remaining segments leaving four closed line segments
- Step 3 removes the middle third of the four remaining segments leaving eight closed line segments.

The process continues **indefinitely**. The set of points in the line segment [0, 1] that are **not** removed during the process is the *Cantor Set.*



(a) **(i)** Complete the table below to show the length of the line segment(s) removed **at each step** for the first 5 steps. Give your answers as fractions.

Step	Step1	Step 2	Step 3	Step 4	Step 5
Length Removed	$\frac{1}{3}$	$\frac{2}{9}$			

(ii) Find the **total** length of all of the line segments removed from the initial line segment of length 1 unit, after a finite number (n) of steps in the process.
Give your answer in terms of n.

(iii) Find the total length removed, from the initial line segment, after an infinite number of steps of the process.

(b) **(i)** Complete the table below to identify the end-points labelled in the diagram.
Give your answers as **fractions.**

Label	A	B	C	D	E	F
End-point						

(ii) Give a reason why $\frac{1}{3}-\frac{1}{9}+\frac{1}{27}-\frac{1}{81}$ is a point in the *Cantor Set*.

(iii) The limit of the series $\frac{1}{3}-\frac{1}{9}+\frac{1}{27}-\cdots$ is a point in the *Cantor Set*. Find this point.

Section B

Q7	Model Solution – 45 Marks	Marking Notes

(a) (i)

	A	B	C	D	E
Fraction	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{4}{27}$	$\frac{8}{81}$	$\frac{16}{243}$

Scale 10C (0, 4, 7, 10)

Low Partial Credit:
- 1 correct fraction given in table
- 1 correct denominator
- 1 correct numerator

High Partial Credit:
- 2 correct fractions given in table
- All numerators correct
- All denominators correct

(a) (ii)

$$a = \frac{1}{3} \quad r = \frac{2}{3}$$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_n = \frac{\frac{1}{3}\left(1 - \left(\frac{2}{3}\right)^n\right)}{1 - \frac{2}{3}}$$

$$S_n = 1 - \left(\frac{2}{3}\right)^n$$

Scale 5C (0, 2, 3, 5)

Low Partial Credit:
- S_n formula with some substitution
- Correct a or correct r identified

High Partial Credit:
- S_n formula fully substituted

(a) (iii)

Infinite Geometric Series $a = \frac{1}{3} \quad r = \frac{2}{3}$

$$S_\infty = \frac{a}{1 - r} = \frac{\frac{1}{3}}{1 - \frac{2}{3}} = 1$$

Or

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(1 - \left(\frac{2}{3}\right)^n\right) = 1$$

Scale 5C (0, 2, 3, 5)

Low Partial Credit:
- S_∞ indicated
- Correct a or correct r identified

High Partial Credit:
- S_∞ fully substituted

Note: If $|r| > 1$, then award low partial credit at most

<table>
<tr>
<td>(b)
(i)</td>
<td>

Label	A	B	C	D	E	F
End-point	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{7}{9}$	$\frac{8}{9}$	$\frac{7}{27}$	$\frac{25}{27}$

</td>
<td>

Scale 10C (0, 4, 7, 10)

Low Partial Credit:

- 1 correct fraction given in table
- All denominators correct

High Partial Credit:

- 4 correct fractions given in table

</td>
</tr>
<tr>
<td>(b)
(ii)</td>
<td>

It is the end point (start point) of a segment

Or

$$\frac{1}{3}-\frac{1}{9}+\frac{1}{27}-\frac{1}{81}=\frac{20}{81}$$

$\frac{6}{27}$ ∘ ∘ $\boldsymbol{E}\,\frac{7}{27}$

$\frac{18}{81}$ $\frac{19}{81}$ $\frac{20}{81}$ $\frac{21}{81}$

Or

$\frac{7}{27}-\frac{1}{81}=\frac{20}{81}$ is a point in the Cantor Set

</td>
<td>

Scale 5B (0, 2, 5)

Mid Partial Credit:

- Relevant but incomplete reason given
- Sum of fractions $=\frac{20}{81}$

</td>
</tr>
<tr>
<td>(b)
(iii)</td>
<td>

$$S_\infty=\frac{a}{1-r}=\frac{\frac{1}{3}}{1-\left(-\frac{1}{3}\right)}=\frac{1}{4}$$

Or

$$\frac{1}{3}+\frac{1}{27}+\frac{1}{243}+\cdots=\frac{\frac{1}{3}}{1-\frac{1}{9}}$$
$$=\frac{3}{8}$$

$$-\left(\frac{1}{9}+\frac{1}{81}+\frac{1}{729}+\cdots\right)=-\left(\frac{\frac{1}{9}}{1-\frac{1}{9}}\right)$$
$$=-\frac{1}{8}$$

$$S_\infty=\frac{3}{8}-\frac{1}{8}$$
$$=\frac{1}{4}$$

</td>
<td>

Scale 10C (0, 4, 7, 10)

Low Partial Credit:

- S_∞ indicated
- S_∞ formula with some substitution
- Correct a or correct r

High Partial Credit:

- S_∞ formula fully substituted

</td>
</tr>
</table>