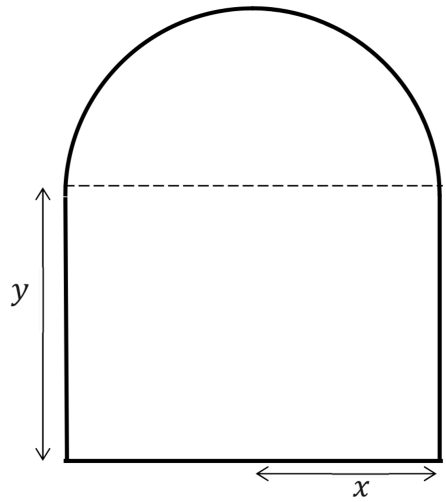
Question 9 **(55 marks)**



Photograph by Lionel Wall.
http://greatenglishchurches.co.uk/html/castle_rising/html

Norman windows consist of a rectangle topped by a semi-circle as shown above.
Let the height **of the rectangle** be y metres and the radius of the semi-circle be x metres as shown. The perimeter of the window is *P*.

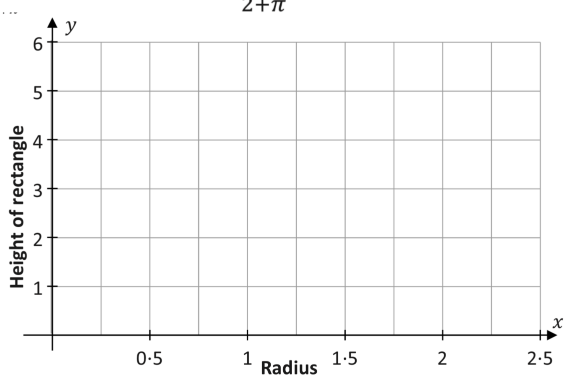
(a) (i) Write *P* in terms of x, y, and π.

(ii) In a particular Norman window the perimeter *P* = 12 metres.
Show that $y = \dfrac{12-(2+\pi)x}{2}$ for $0 \leq x \leq \frac{12}{2+\pi}$ where $x \in \mathbb{R}$.

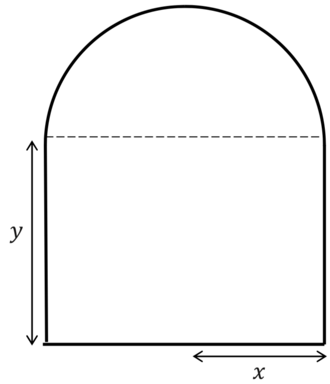
(b) (i) Complete the table on the right.

x	0	$\dfrac{12}{2+\pi}$
$y = \dfrac{12-(2+\pi)x}{2}$		

(ii) On the diagram below, draw the graph of the linear function, $y = \dfrac{12-(2+\pi)x}{2}$ for $0 \leq x \leq \frac{12}{2+\pi}$ where $x \in \mathbb{R}$.



(c) (i) The Norman window shown below has a perimeter of 12 metres and $y = \dfrac{12-(2+\pi)x}{2}$.



Show that the function $a(x) = \dfrac{24x-(\pi+4)x^2}{2}$ represents the area of the window, in terms of x and π.

(ii) Find $a'(x)$.

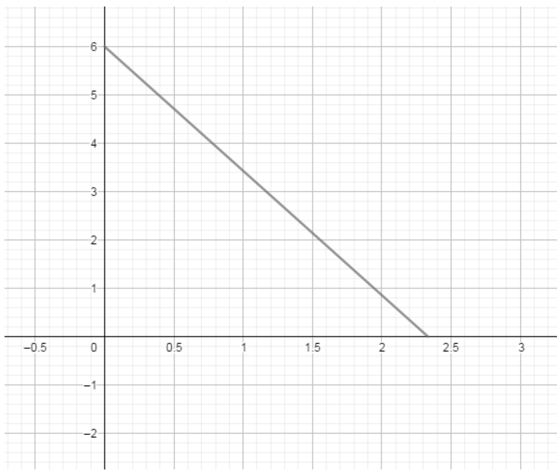
(iii) Find the relationship between x and y when the area of the window in **part (c)(i)** is at its maximum.

Q9	Model Solution – 55 Marks	Marking Notes
(a) (i)	$= 2(x) + 2(y) + \frac{1}{2}(2\pi)(x)$ $= 2x + 2y + \pi x$	**Scale 5C (0, 2, 3, 5)** *Low Partial Credit:* - Some relevant substitution into perimeter formula - Circumference of circle of radius x found i.e. $2\pi x$ *High Partial Credit:* - Two of the three terms found
(a) (ii)	$2x + 2y + \pi x = 12$ $2y = 12 - 2x - \pi x$ $y = \frac{12 - 2x - \pi x}{2}$ $y = \frac{12 - (2 + \pi)x}{2}$	**Scale 5C (0, 2, 3, 5)** *Low Partial Credit:* - Some relevant substitution into equation *High Partial Credit:* - y term isolated correctly in equation <u>Note</u>: Accept candidates answer from (a)(i) provided it doesn't oversimplify the work. <u>Note</u>: Must draw a relevant conclusion from incorrect work

(b) (i) + (ii)

Table and Graph

x	0	$\frac{12}{2+\pi}$
$y = \frac{12-(2+\pi)x}{2}$	6	0



Scale 15D (0, 4, 7, 11, 15)

Low Partial Credit:
- One correct table entry
- One correct plot of incorrect point

Mid Partial Credit:
- 2 table entries correct
- 2 incorrect points plotted and joined

High Partial Credit:
- 2 correct points plotted but not joined with correct table entries

Full Credit −1:
- Two correct points plotted and joined but the function is not graphed in the stated domain

<u>Note</u>: Accept $2{\cdot}25 \leq x \leq 2{\cdot}5$ for x–intercept

<table>
<tr><td>(b)
(iii)</td><td>$y = \dfrac{12-(2+\pi)x}{2}$
$y = 6 - \left(\dfrac{2+\pi}{2}\right)x$

$m = -\left(\dfrac{2+\pi}{2}\right)$

$m = -2{\cdot}57$

Or

$m = \dfrac{0-6}{\frac{12}{2+\pi} - 0}$

$m = -\left(\dfrac{2+\pi}{2}\right)$

$m = -2{\cdot}57$

Intepretation:
For each 1m rise in the radius of the semi-circle, the height of the rectangle falls by approximately 2·57 m</td><td>.
Scale 5C (0, 2, 3, 5)
Low Partial Credit:
- Some substitution into slope formula
- Slope isolated in the equation of the line formula
- $\frac{dy}{dx}$
- $\frac{rise}{run}$ with some relevant substitution
- Some effort at differentiation

High Partial Credit:
- Slope found

<u>Note</u>: Accept $-2{\cdot}7 \leq$ slope $\leq -2{\cdot}5$ from relevant work</td></tr>
<tr><td>(c)
(i)</td><td>$a = 2xy + \dfrac{\pi x^2}{2}$
$= \dfrac{2x[(12-(2+\pi)x]}{2} + \dfrac{\pi x^2}{2}$
$= \dfrac{24x - 4x^2 - 2\pi x^2}{2} + \dfrac{\pi x^2}{2}$
$= \dfrac{24x - (\pi+4)x^2}{2}$</td><td>0.
Scale 5C (0, 2, 3, 5)
Low Partial Credit:
- area of rectangle correct
- area of semi-circle correct

High Partial Credit:
- Both areas correct in terms of x and added</td></tr>
</table>

(c) (ii)	$a(x) = \frac{1}{2}(24x - (\pi + 4)x^2)$ $a'(x) = \frac{1}{2}(24 - 2(\pi + 4)x)$ $= 12 - (\pi + 4)x$	**Scale 5B (0, 2, 5)** *Mid Partial Credit:* - Some correct differentiation
(c) (iii)	$a'(x) = 0$ $12 - (\pi + 4)x = 0$ $(\pi + 4)x = 12$ $x = \frac{12}{\pi+4}$ (1·68) $y = \frac{12-(2+\pi)x}{2}$ $\left(= \frac{12-(5\cdot14)\ 1\cdot68}{2} \approx 1\cdot68\right)$ $= \frac{12 - (2+\pi)(\frac{12}{\pi+4})}{2}$ $= \frac{12(\pi+4) - (2+\pi)(12)}{2(\pi+4)}$ $= \frac{12\pi + 48 - 24 - 12\pi}{2(\pi+4)}$ $= \frac{24}{2(\pi+4)}$ $= \frac{12}{\pi+4}$ $= x$ Area Max when height equals the radius	**Scale 15D (0, 4, 7, 11, 15)** *Low Partial Credit:* - $a'(x)$ used - States $\frac{dy}{dx} = 0$ *Mid Partial Credit* - Value of x at maximum found *High Partial Credit:* - Value of y at maximum fully substituted