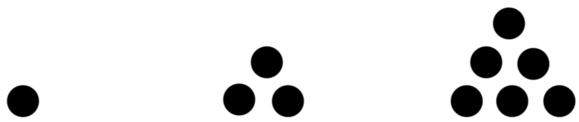
Question 7 **(50 marks)**

(a) A number of the form $1 + 2 + 3 + \cdots + n$ is sometimes called a **triangular number** because it can be represented as an equilateral triangle.
The diagram below shows the first three terms in the sequence of triangular numbers.



$T_1 = 1$ $\quad T_2 = 1 + 2 = 3$ $\quad T_3 = 1 + 2 + 3 = 6$

(i) Complete the table below to list the next five triangular numbers.

Term	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
Triangular Number	1	3	6					

(ii) The n^{th} triangular number can be found directly using the formula

$$T_n = \frac{n(n+1)}{2}.$$

Is 1275 a triangular number? Give a reason for your answer.

(b) **(i)** The $(n+1)^{\text{th}}$ triangular number can be written as $T_{n+1} = T_n + (n+1)$, where $n \in \mathbb{N}$.
Write the expression $\frac{n(n+1)}{2} + (n+1)$ as a single fraction in its simplest form.

(ii) Prove that the **sum** of any two consecutive triangular numbers will **always** be a square number (a number in the form k^2, where $k \in \mathbb{N}$).

(iii) Two consecutive triangular numbers **sum** to 12 544.
Find the smaller of these two numbers.

(c) Some numbers are both triangular and square, for example 36.
Leonhard Euler (1778) discovered the following formula for these numbers

$$N_k = \left(\frac{\left(3 + 2\sqrt{2}\right)^k - \left(3 - 2\sqrt{2}\right)^k}{4\sqrt{2}}\right)^2$$

where N_k is the k^{th} number that is both triangular and square.
Use Euler's formula to find N_3, the third number that is both triangular and square.

(d) Prove using **induction** that, for all $n \in \mathbb{N}$, the sum of the first n square numbers can be found using the formula:

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

<table>
<tr><th>Q7</th><th>Model Solution – 50 Marks</th><th>Marking Notes</th></tr>
<tr><td>(a)
(i)</td><td>
<table>
<tr><td>T.</td><td>T_1</td><td>T_2</td><td>T_3</td><td>T_4</td><td>T_5</td><td>T_6</td><td>T_7</td><td>T_8</td></tr>
<tr><td>No.</td><td>1</td><td>3</td><td>6</td><td>10</td><td>15</td><td>21</td><td>28</td><td>36</td></tr>
</table>
</td><td>Scale 10C (0, 4, 8, 10)
Low Partial Credit:
One correct new entry

High Partial Credit:
Three correct new entries</td></tr>
<tr><td>(a)
(ii)</td><td>$\frac{n}{2}(n+1) = 1275$
$n^2 + n - 2250 = 0$
$(n-50)(n+51)$
$n = 50$
1275 is the 50th triangular number</td><td>Scale 5C (0, 3, 4, 5)
Low Partial Credit:
$\frac{n}{2}(n+1) = 1275$

High Partial Credit:
$n = 50$
Note: accept T_{50} as valid reason</td></tr>
<tr><td>(b)
(i)</td><td>$T_{n+1} = T_n + (n+1)$
$= \frac{n}{2}(n+1) + (n+1)$
$= \frac{n(n+1) + 2(n+1)}{2}$
$= \frac{(n+1)(n+2)}{2}$</td><td>Scale 5C (0, 3, 4, 5)
Low Partial Credit:
2 identified as C.D.
Correct numerator

High Partial Credit:
$= \frac{n(n+1) + 2(n+1)}{2}$</td></tr>
<tr><td>(b)
(ii)</td><td>$T_{n+1} + T_n$
$= \frac{(n+1)(n+2)}{2} + \frac{n}{2}(n+1)$
$= \frac{(n+1)(2n+2)}{2}$
$= \frac{2(n+1)(n+1)}{2}$
$= (n+1)^2$</td><td>Scale 5C (0, 3, 4, 5)
Low Partial Credit:
$T_{n+1} + T_n$ with some substitution
Particular case verification

High Partial Credit:
$\frac{(n+1)(n+2)}{2} + \frac{n}{2}(n+1)$</td></tr>
</table>

(b) (iii)	$$(n+1)^2 = 12544$$ $$n+1 = \sqrt{12544} = 112$$ $$n = 111$$ $$n = 111$$ T_{111} is the smaller term $$T_{111} = \frac{111(112)}{2}$$ $$T_{111} = 6216$$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* $(n+1)^2$ *High Partial Credit:* $n = 111$, or $n = 112$
(c)	$$N_3 = \left(\frac{(3+2\sqrt{2})^3 - (3-2\sqrt{2})^3}{4\sqrt{2}}\right)^2$$ $$= 1225$$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* Formula with some substitution *High Partial Credit:* Formula fully substituted *Full Credit:* Correct answer with no work shown

(d)	$1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ $\boldsymbol{P(1)}$: $1 = \frac{1(2)(3)}{6}$ $\boldsymbol{P(k)}$: $1 + 4 + 9 + \cdots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$ $\boldsymbol{P(k+1)}$: $1 + 4 + 9 + \cdots + k^2 + (k+1)^2 = \dfrac{(k+1)(k+2)(2k+3)}{6}$ $LHS = \dfrac{k(k+1)(2k+1)}{6} + (k+1)^2$ $LHS = \dfrac{k(k+1)(2k+1) + 6(k+1)^2}{6}$ $LHS = \dfrac{(k+1)[k(2k+1) + 6(k+1)]}{6}$ $LHS = \dfrac{(k+1)[2k^2 + 7k + 6]}{6}$ $\dfrac{(k+1)(k+2)(2k+3)}{6} = RHS$ Thus the proposition is true for $n = k + 1$ provided it is true for $n = k$ but it is true for $n = 1$ and therefore true for all positive integers.	**Scale 15D (0, 4, 7, 11, 15)** *Low Partial Credit:* Step $P(1)$ *Mid Partial Credit:* Step $P(k+1)$ *High Partial Credit:* Uses *Step* $P(k)$ to prove *Step* $P(k+1)$ *Full Credit(–1):* Concluding statement missing **Note:** Accept *Step* $P(1)$, *Step* $P(k)$, *Step* $P(k+1)$ in any order