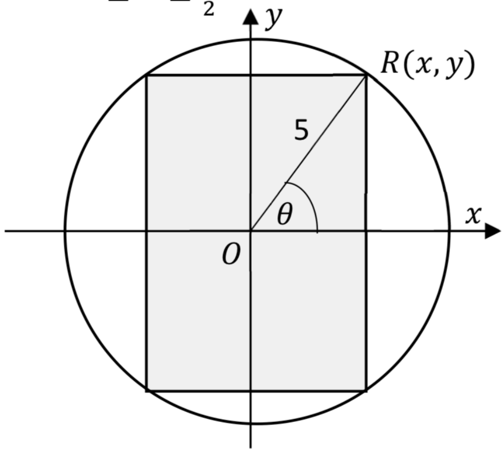
Question 8 **(45 marks)**

A rectangle is inscribed in a circle of radius 5 units and centre $O(0, 0)$ as shown below.
Let $R(x, y)$, where $x, y \in \mathbb{R}$, be the vertex of the rectangle in the first quadrant as shown.
Let θ be the angle between $[OR]$ and the positive x-axis, where $0 \leq \theta \leq \frac{\pi}{2}$.



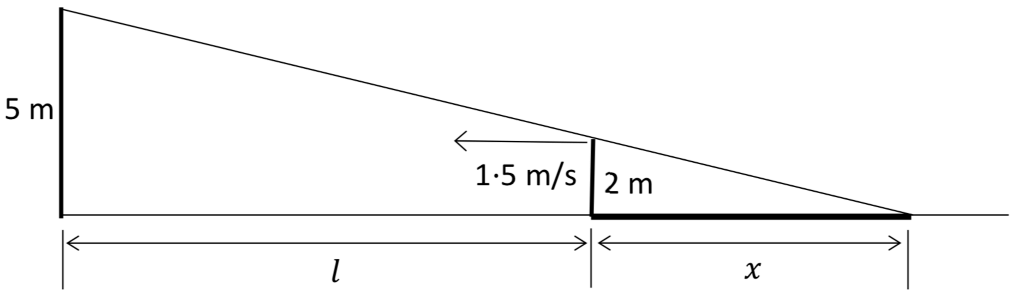
(a) **(i)** The point $R(x, y)$ can be written as $(a \cos \theta, b \sin \theta)$, where $a, b \in \mathbb{R}$.
Find the value of a and the value of b.

(ii) Show that $A(\theta)$, the area of the rectangle, measured in square units, can be written as $A(\theta) = 50 \sin 2\theta$.

(iii) Use calculus to show that the rectangle with maximum area is a square.

(iv) Find this maximum area.

(b) A person who is 2 m tall is walking towards a streetlight of height 5 m at a speed of 1·5 m/s. Find the rate, in m/s, at which the length of the person's shadow (x), cast by the streetlight, is changing.



Q8	Model Solution – 45 Marks	Marking Notes
(a) (i)	$\cos\theta = \frac{x}{5}$ $\quad \sin\theta = \frac{y}{5}$ $5\cos\theta = x$ $\quad 5\sin\theta = y$ $(x, y) = (5\cos\theta\,,\ 5\sin\theta)$ $\therefore a = 5,\ \ b = 5$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* $\cos\theta = \frac{x}{5}$ or equivalent *High Partial Credit:* a or b found Correct answer without work
(a) (ii)	$A(\theta) = (10\cos\theta) \times (10\sin\theta)$ $A(\theta) = 100\cos\theta\sin\theta$ $= 50 \times 2\cos\theta\sin\theta$ $= 50(\sin 2\theta)$	**Scale 10C (0, 4, 8, 10)** *Low Partial Credit:* xy $(10\cos\theta) \times (10\sin\theta)$ *High Partial Credit:* $100\cos\theta\sin\theta$
(a) (iii)	$A(\theta) = 50\sin 2\theta$ $A'(\theta) = 50\cos 2\theta \times 2$ $A'(\theta) = 100\cos 2\theta = 0$ $\cos 2\theta = 0$ $2\theta = \frac{\pi}{2}$ $\theta = \frac{\pi}{4}$ $2x = 2\left(5\cos\left(\frac{\pi}{4}\right)\right) = 5\sqrt{2}$ $2y = 2\left(5\sin\left(\frac{\pi}{4}\right)\right) = 5\sqrt{2}$ $\Longrightarrow$ Square	**Scale 15D (0, 4, 7, 11, 15)** *Low Partial Credit:* $a'(\theta)$ States $\frac{dy}{dx} = 0$ *Mid Partial Credit:* Correct differentiation *High Partial Credit:* Value of θ at maximum found Value of x or y at maximum fully substituted *No Credit:* No differentiation
(a) (iv)	Max area $= 5\sqrt{2} \times 5\sqrt{2}$ $= 50$ Square units **Or** Max area $= 50(\sin 2\theta)$ $50(\sin\frac{\pi}{2})$ $= 50$ Square units	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* xy length $\times$ width $50(\sin 2\theta)$ *High Partial Credit:* Area formula fully substituted

(b)	$\frac{dx}{dt} = \frac{dx}{dl} \cdot \frac{dl}{dt}$ $\frac{2}{5} = \frac{x}{l + x}$ $2l + 2x = 5x$ $x = \frac{2}{3}l$ $\frac{dx}{dl} = \frac{2}{3}$ $\frac{dx}{dt} = \frac{2}{3} \times \frac{3}{2}$ $\frac{dx}{dt} = 1$ m/sec	**Scale 10D (0, 3, 5, 8, 10)** *Low Partial Credit:* $\frac{dx}{dt}$ or $\frac{dx}{dl}$ or $\frac{dl}{dt}$ given Reference to similar triangles $\frac{2}{5}$ or $\frac{5}{2}$ *Mid Partial Credit:* $\frac{dx}{dt} = \frac{dx}{dl} \cdot \frac{dl}{dt}$ or equivalent with one relevant substitution $x = \frac{2}{3}l$ *High Partial Credit:* $\frac{dx}{dl}$ and $\frac{dl}{dt}$ found