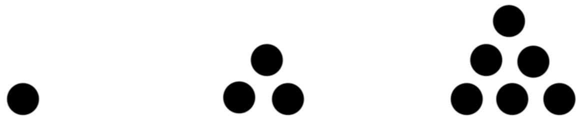
Question 7 **(50 marks)**

(a) A number of the form $1 + 2 + 3 + \cdots + n$ is sometimes called a **triangular number** because it can be represented as an equilateral triangle.
The diagram below shows the first three terms in the sequence of triangular numbers.



$T_1 = 1$ $\quad$ $T_2 = 1 + 2 = 3$ $\quad$ $T_3 = 1 + 2 + 3 = 6$

(i) Complete the table below to list the next five triangular numbers.

Term	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
Triangular Number	1	3	6					

(ii) The n^{th} triangular number can be found directly using the formula

$$T_n = \frac{n(n+1)}{2}.$$

Is 1275 a triangular number? Give a reason for your answer.

(b) **(i)** The $(n+1)^{\text{th}}$ triangular number can be written as $T_{n+1} = T_n + (n+1)$, where $n \in \mathbb{N}$.
Write the expression $\frac{n(n+1)}{2} + (n+1)$ as a single fraction in its simplest form.

(ii) Prove that the **sum** of any two consecutive triangular numbers will **always** be a square number (a number in the form k^2, where $k \in \mathbb{N}$).

(iii) Two consecutive triangular numbers **sum** to 12 544.
Find the smaller of these two numbers.

(c) Some numbers are both triangular and square, for example 36.
Leonhard Euler (1778) discovered the following formula for these numbers

$$N_k = \left(\frac{\left(3+2\sqrt{2}\right)^k - \left(3-2\sqrt{2}\right)^k}{4\sqrt{2}}\right)^2$$

where N_k is the k^{th} number that is both triangular and square.
Use Euler's formula to find N_3, the third number that is both triangular and square.

(d) Prove using **induction** that, for all $n \in \mathbb{N}$, the sum of the first n square numbers can be found using the formula:

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$