Parametric Differentiation

Exercise 7C

Remember: $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$

Q1 Find $\frac{dy}{dx}$.

$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$

(i) $x = 3t + 1$, $y = t^2$

$\frac{dx}{dt} = 3$, $\frac{dy}{dt} = 2t$, $\frac{dy}{dx} = \frac{2t}{3}$

Q1 Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

(ii) $x = 2t + 1$, $y = t^2 - 1$

$\frac{dx}{dt} = 2$ $\quad$ $\frac{dy}{dt} = 2t$ $\quad$ $\frac{dy}{dx} = \frac{2t}{2} = t$

Q1 Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

(iii) $x = t^3 + t$ $\quad$ $y = 2t^2$

$\frac{dx}{dt} = 3t^2 + 1$ $\quad$ $\frac{dy}{dt} = 4t$ $\quad$ $\frac{dy}{dx} = \frac{4t}{3t^2+1}$

Q1 Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

(iv) $x = 5t$ $\quad y = t - \frac{1}{t} = t - t^{-1}$

$\frac{dx}{dt} = 5$ $\quad \frac{dy}{dt} = 1 + t^{-2} = 1 + \frac{1}{t^2}$

$$\frac{dy}{dx} = \frac{1 + \frac{1}{t^2}}{5} = \frac{t^2+1}{5t^2}$$

Q2 $x = 1 - t^2$ $\quad y = t(1-t^2) = t - t^3$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

Show $\frac{dy}{dx} = \frac{3t^2-1}{2t}$

$\frac{dx}{dt} = -2t$ $\quad \frac{dy}{dt} = 1 - 3t^2$

$$\frac{dy}{dx} = \frac{-3t^2+1}{-2t} = \frac{3t^2-1}{2t} \quad \text{QED}$$

Q3 $\quad x = \dfrac{t}{1-t} \qquad y = t^2 \qquad$ Find $\dfrac{dy}{dx}(t=2)$.

$$\boxed{\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}}$$

$\dfrac{dx}{dt} = ?$ QUOTIENT

$u = t \qquad v = 1-t$

$\dfrac{du}{dt} = 1 \qquad \dfrac{dv}{dt} = -1$

$$\frac{dx}{dt} = \frac{(1-t)(1) - (t)(-1)}{(1-t)^2}$$

$$= \frac{1-t+t}{(1-t)^2} = \frac{1}{(1-t)^2}$$

$\dfrac{dy}{dt} = 2t$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{2t}{1} \cdot \frac{(1-t)^2}{1}$$

$$\frac{dy}{dx}(t=2) = 2(2)(1-2)^2$$

$$= 4$$

Q4 $\quad$ Find Slope at $t = \frac{1}{3}$

$x = 2t^2 - 1 \qquad y = 3t^2 + t$

$$\boxed{\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}}$$

$\dfrac{dx}{dt} = 4t \qquad \dfrac{dy}{dt} = 6t + 1$

$$\frac{dy}{dx} = \frac{6t+1}{4t}$$

$$\frac{dy}{dx}(t=\tfrac{1}{3}) = \frac{6(\frac{1}{3})+1}{4(\frac{1}{3})} = \frac{3}{(\frac{4}{3})} = \frac{9}{4}$$

Q5 Find equation of tangent to
$x = t^2,\ y = 4t$ at $t = -1$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$\frac{dx}{dt} = 2t$ $\frac{dy}{dt} = 4$ $\frac{dy}{dx} = \frac{4}{2t} = \frac{2}{t}$

$\left.\frac{dy}{dt}\right|_{(t=-1)} = \frac{2}{-1} = -2$ [this is slope of tangent]

need point on curve? $x = t^2 = (-1)^2 = 1$, $y = 4t = 4(-1) = -4$
pt $(1, -4)$

equation: $y - y_1 = m(x - x_1) \Rightarrow y + 4 = -2(x - 1) \Rightarrow 2x + y + 2 = 0$

Q6 Find equation of tangent to
$x = t^2,\ y = t + \frac{2}{t}$ where $t = 1$.

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$\frac{dx}{dt} = 2t$ $\frac{dy}{dt} = 1 - 2t^{-2}$

$\frac{dy}{dx} = \frac{1 - 2t^{-2}}{2t} \Rightarrow \left.\frac{dy}{dx}\right|_{(t=1)} = \frac{1 - 2(1)^{-2}}{2(1)} = \frac{1-2}{2} = -\frac{1}{2}$ slope

When $t = 1 \Rightarrow x = (1)^2 = 1$, $y = 1 + \frac{2}{1} = 3 \Rightarrow$ pt $(1, 3)$

Equation: $y - y_1 = m(x - x_1)$
$y - 3 = -\frac{1}{2}(x - 1) \Rightarrow 2y - 6 = -x + 1$
$\Rightarrow x + 2y - 7 = 0$

Q7 $x = 2\cos\theta \qquad y = 3\sin\theta$

$$\frac{dy}{dx}\left(\theta=\frac{\pi}{4}\right) = ?$$

$$\boxed{\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}}$$

$$\frac{dx}{d\theta} = -2\sin\theta \qquad \frac{dy}{d\theta} = 3\cos\theta$$

$$\frac{dy}{dx} = \frac{3\cos\theta}{-2\sin\theta}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\frac{dy}{dx}\left(\theta=\frac{\pi}{4}\right) = \frac{3\cos\left(\frac{\pi}{4}\right)}{-2\sin\left(\frac{\pi}{4}\right)} = \frac{\left(\frac{3}{\sqrt{2}}\right)}{-\left(\frac{2}{\sqrt{2}}\right)} = \frac{-3}{2}$$

Q8 $x = \sin^2 t \qquad y = \cos t$

$$\frac{dy}{dx}\left(t=\frac{\pi}{3}\right) = ?$$

$$\boxed{\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}}$$

$$\frac{dx}{dt} = 2\sin t\,(\cos t) \qquad \frac{dy}{dt} = -\sin t$$

$$\cos\frac{\pi}{3} = \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{-\cancel{\sin t}\,1}{2\cancel{\sin t}\cos t} = \frac{-1}{2\cos t}$$

$$\frac{dy}{dx}\left(t=\frac{\pi}{3}\right) = \frac{-1}{2\cos\frac{\pi}{3}} = \frac{-1}{2\left(\frac{1}{2}\right)} = -1$$

Q9 $x = 2\sin\theta \qquad y = 2 + \cos 2\theta$

$$\frac{dy}{dx}\left(\theta = \frac{\pi}{4}\right) = ?$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$\frac{dx}{d\theta} = 2\cos\theta \qquad \frac{dy}{d\theta} = -2\sin 2\theta$$

$$\frac{dy}{dx} = \frac{-2\sin\theta}{2\cos\theta} = -\tan\theta$$

$$\frac{dy}{dx}\left(\theta = \frac{\pi}{4}\right) = -\tan\frac{\pi}{4} = -1$$

$\tan\frac{\pi}{4} = 1$

Q10 $x = 4\cos\theta + 3\sin\theta$

$y = 3\cos\theta - 4\sin\theta \qquad -\pi < \theta < \pi$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$\frac{dy}{dx}\left(\theta = \frac{\pi}{2}\right) = ?$$

$$\frac{dx}{d\theta} = -4\sin\theta + 3\cos\theta \qquad \frac{dy}{d\theta} = -3\sin\theta - 4\cos\theta$$

$$\frac{dy}{dx} = \frac{-3\sin\theta - 4\cos\theta}{-4\sin\theta + 3\cos\theta}$$

$$\frac{dy}{dx}\left(\theta = \frac{\pi}{2}\right) = \frac{-3(1) - 4(0)}{-4(1) + 3(0)} = \frac{3}{4}$$

$\sin\frac{\pi}{2} = 1$

$\cos\frac{\pi}{2} = 0$

Q11 $x = 1 - \cos\theta \qquad y = \theta - \sin\theta$

$\frac{dy}{dx}(\theta = \frac{\pi}{3}) = ?$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$\frac{dx}{d\theta} = +\sin\theta \qquad \frac{dy}{d\theta} = 1 - \cos\theta$

$\cos\frac{\pi}{3} = \frac{1}{2}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\frac{dy}{dx} = \frac{1-\cos\theta}{\sin\theta}$

$\frac{dy}{dx}(\theta = \frac{\pi}{3}) = \frac{1 - \frac{1}{2}}{\sqrt{3}/2} = \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Q12 Find tangent at $t = 2$

$x = t^2 + 2 \quad , \quad y = t^3 - 1$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 3t^2 \qquad \frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2}$

$\frac{dy}{dx}(t=2) = \frac{3(2)}{2} = 3$ slope

point when $t = 2$? $x = (2)^2 + 2 = 6$, $y = (2)^3 - 1 = 7$, pt $(6,7)$

equation:

$y - y_1 = m(x - x_1)$

$y - 7 = 3(x - 6)$

$y - 7 = 3x - 18$

$3x - y - 11 = 0$

Q13 $x = 2t - \sin 2t$, $y = 4\cos t$, $0 < t < \frac{\pi}{2}$

Use $\cos 2t = 1 - 2\sin^2 t$ to show that $\frac{dy}{dx} = -\frac{1}{\sin t}$

$$\frac{dx}{dt} = 2 - 2\cos 2t \qquad \frac{dy}{dt} = -4\sin t$$

$$\frac{dy}{dx} = \frac{-4\sin t}{2 - 2\cos 2t} = \frac{-4\sin t}{2 - 2(1 - 2\sin^2 t)} = \frac{-4\sin t}{\cancel{2} - \cancel{2} + 4\sin^2 t}$$

$$\frac{-\cancel{4}\sin t}{\cancel{4}\sin^2 t} = \frac{-1}{\sin t} \quad \text{QED}$$

$$\boxed{\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}}$$

Q14

$x = \sqrt{3}\tan\theta$, $y = \sqrt{3}\cos\theta$, $0 \le \theta \le \pi$

(i) Find $\frac{dy}{d\theta}$ (ii) Equation of tangent at $\theta = \pi/6$

$$\frac{dx}{d\theta} = \sqrt{3}\sec^2\theta \qquad \frac{dy}{d\theta} = -\sqrt{3}\sin\theta$$

$$\frac{dy}{d\theta} = \frac{-\cancel{\sqrt{3}}\sin\theta}{\cancel{\sqrt{3}}\sec^2\theta} = -\sin\theta\cos^2\theta$$

$$\frac{dy}{d\theta}(\theta = \pi/6) = -\left(\tfrac{1}{2}\right)\left(\tfrac{\sqrt{3}}{2}\right)^2 = -\left(\tfrac{1}{2}\right)\left(\tfrac{3}{4}\right) = -\tfrac{3}{8} \quad \text{slope}$$

$$\theta = \pi/6 \Rightarrow x = \sqrt{3}\left(\tfrac{1}{\sqrt{3}}\right) = 1 \;,\; y = \sqrt{3}\left(\tfrac{\sqrt{3}}{2}\right) = \tfrac{3}{2}$$

equation: $y - y_1 = m(x - x_1)$ $\rightarrow$ $y - \frac{3}{2} = -\frac{3}{8}(x - 1)$

$$8y - 12 = -3x + 3 \Rightarrow 3x + 8y - 15 = 0$$

note:

Since: $\sec^2\theta = \frac{1}{\cos^2\theta}$

$\Rightarrow \frac{1}{\sec^2\theta} = \cos^2\theta$

$\sin\frac{\pi}{6} = \frac{1}{2}$

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

Q15

$x = \cos\theta + \theta\sin\theta \qquad y = \sin\theta - \theta\cos\theta$

Show $\frac{dy}{dx} = \tan\theta$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$\frac{dx}{dt} = \cancel{-\sin\theta} + \theta\cos\theta + \cancel{\sin\theta} = \theta\cos\theta$

$\frac{dy}{dt} = \cancel{\cos\theta} + \theta\sin\theta - \cancel{\cos\theta} = \theta\sin\theta$

$\frac{dy}{d\theta} = \frac{\cancel{\theta}\sin\theta}{\cancel{\theta}\cos\theta} = \tan\theta$ QED

$\theta\sin\theta$ is Product

$u = \theta \qquad v = \sin\theta$

$\frac{du}{d\theta} = 1 \qquad \frac{dv}{d\theta} = \cos\theta$

derivative = $\theta\cos\theta + 1\sin\theta$

$\theta\cos\theta$ is Product

$u = \theta \qquad v = \cos\theta$

$\frac{du}{d\theta} = 1 \qquad \frac{dv}{d\theta} = -\sin\theta$

Derivative = $-\theta\sin\theta + 1\cos\theta$

Q16 $x = \cos^3 t$, $y = \sin^3 t$ (i) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$

(ii) If $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{a}{b}(\sin 2t)^2$, find a and b ?

$\frac{dx}{dt} = -3\cos^2 t\sin t \quad , \quad \frac{dy}{dt} = 3\sin^2 t\cos t$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t$

$= 9\cos^2 t\sin^2 t(\cos^2 t + \sin^2 t)$

$= 9\cos^2 t\sin^2 t$

$= 9(\cos t\sin t)^2$

$= 9\left(\frac{\sin 2t}{2}\right)^2 = \frac{9}{4}(\sin 2t)^2 \Rightarrow a = 9, b = 4$

note:

$\cos^2 t + \sin^2 t = 1$

and

$\sin 2t = 2\sin t\cos t$

$\Rightarrow \sin t\cos t = \frac{\sin 2t}{2}$