Differentiation of inverse trigonometric functions

$$y = \tan^{-1}\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{a}{a^2 + x^2}$$

$$y = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$$

Exercise 7D

Q1 (i) $y = \sin^{-1}(6x) = \sin^{-1}\left(\frac{6x}{1}\right)$

$$y = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$$

CHAIN RULE

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1^2 - (6x)^2}}\right)(6)$$

$$= \frac{6}{\sqrt{1 - 36x^2}}$$

Q1 (ii)

$y = \tan^{-1}(3x) = \tan^{-1}\left(\frac{3x}{1}\right)$

CHAIN RULE

$\frac{dy}{dx} = \left(\frac{1}{1^2 + (3x)^2}\right)(3)$

$= \frac{3}{1 + 9x^2}$

$y = \tan^{-1}\left(\frac{x}{a}\right)$

$\frac{dy}{dx} = \frac{a}{a^2 + x^2}$

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Q1 (iii)

$y = \sin^{-1}\left(\frac{2x+1}{1}\right)$

$y = \sin^{-1}\frac{x}{a} \quad \frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$

$\frac{dy}{dx} = \frac{1\,(2)}{\sqrt{(1)^2 - (2x+1)^2}} = \frac{2}{\sqrt{1 - [4x^2 + 4x + 1]}}$

$= \frac{2}{\sqrt{\cancel{1} - 4x^2 - 4x - \cancel{1}}} = \frac{2}{\sqrt{-4x^2 - 4x}}$

Q1 (iv)

$$y = \tan^{-1}(x^2) = \tan^{-1}\left(\frac{x^2}{1}\right)$$

CHAIN RULE

$$\frac{dy}{dx} = \left(\frac{1}{1^2 + (x^2)^2}\right)(2x)$$

$$= \frac{2x}{1 + x^4}$$

$$y = \tan^{-1}\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{a}{a^2 + x^2}$$

Q2

$$y = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$$

$$y = \sin^{-1}(3x - 1) = \sin^{-1}\left(\frac{3x-1}{1}\right)$$

CHAIN RULE

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1^2 - (3x-1)^2}}\right)(3)$$

$$= \frac{3}{\sqrt{1 - [9x^2 - 6x + 1]}} = \frac{3}{\sqrt{\cancel{1} - 9x^2 + 6x - \cancel{1}}}$$

$$= \frac{3}{\sqrt{6x - 9x^2}}$$ QED

Q3 (i) $f(x) = \sin^{-1}(2x)$ Find $f'(0)$?

$$y = \sin^{-1}\left(\frac{x}{a}\right)$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$$

$$f(x) = \sin^{-1}\left(\frac{2x}{1}\right)$$

CHAIN RULE $$f'(x) = \left(\frac{1}{\sqrt{1^2 - (2x)^2}}\right)(2) = \frac{2}{\sqrt{1 - 4x^2}}$$

$$f'(0) = \frac{2}{\sqrt{1 - 4(0)^2}} = 2$$

Q3 (ii) $f(x) = \tan^{-1}(4x)$, $f'\left(\frac{1}{4}\right) = ?$

$$y = \tan^{-1}\left(\frac{x}{a}\right)$$
$$\frac{dy}{dx} = \frac{a}{a^2 + x^2}$$

$$f(x) = \tan^{-1}\left(\frac{4x}{1}\right)$$

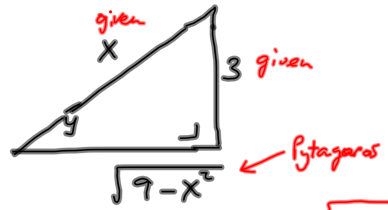
$$f'(x) = \left(\frac{1}{1^2 + (4x)^2}\right)(4) = \frac{4}{1 + 16x^2}$$

$$f'\left(\frac{1}{4}\right) = \frac{4}{1 + 16\left(\frac{1}{4}\right)^2} = 2$$

Q4 (i) Use implicit differentiation to find $\frac{dy}{dx}$

$$y = \sin^{-1}\left(\frac{3}{x}\right)$$

$$\Rightarrow \sin y = \frac{3}{x} \xrightarrow{\text{DRAW TRIANGLE}}$$



$$\cos y = \frac{\sqrt{9-x^2}}{x}$$

$$\sin y = 3x^{-1}$$

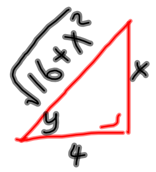
differentiate $\Rightarrow$ $\cos y \frac{dy}{dx} = -3x^{-2}$

$$\frac{dy}{dx} = \frac{-3x^{-2}}{\cos y} = \frac{-3}{x^2\left(\frac{\sqrt{9-x^2}}{x}\right)} = \frac{-3}{x\sqrt{9-x^2}}$$

Q4 (ii) Use implicit differentiation to find $\frac{dy}{dx}$

$$y = \tan^{-1}\left(\frac{x}{4}\right)$$

$$\tan y = \frac{x}{4} \xrightarrow{\text{draw triangle}}$$



$$\cos y = \frac{4}{\sqrt{16+x^2}}$$

$$(\cos y)^2 = \frac{16}{16+x^2}$$

$$\sec^2 y \frac{dy}{dx} = \frac{1}{4}$$

$$\frac{dy}{dx} = \frac{1}{4\sec^2 y} = \frac{\cos^2 y}{4} = \frac{1}{4}\left(\frac{16}{16+x^2}\right) = \frac{4}{16+x^2}$$

Q5

(i) $y = x\sin^{-1}x$

PRODUCT

$u = x \qquad v = \sin^{-1}\left(\frac{x}{1}\right)$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$

$$\frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x$$

$y = \sin^{-1}\left(\frac{x}{a}\right)$

$\frac{dy}{dx} = \frac{1}{\sqrt{a^2-x^2}}$

Q5 (ii) $y = 2x\tan^{-1}x$

PRODUCT

$u = 2x \qquad v = \tan^{-1}\left(\frac{x}{1}\right)$

$\frac{du}{dx} = 2 \qquad \frac{dv}{dx} = \frac{1}{1+x^2}$

$$\frac{dy}{dx} = \frac{2x}{1+x^2} + 2\tan^{-1}x$$

$y = \tan^{-1}\left(\frac{x}{a}\right)$

$\frac{dy}{dx} = \frac{a}{a^2+x^2}$

Q6 $y = \left(\sin^{-1}\frac{x}{1}\right)^2$

$y = \sin^{-1}\left(\frac{x}{a}\right)$

$\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$

using chain rule

$\frac{dy}{dx} = 2(\sin^{-1}x)\left(\frac{1}{\sqrt{1 - x^2}}\right)$ ans

Q7 $y = \sin^{-1}(\cos x)$ $\frac{dy}{dx} = ?$

$y = \sin^{-1}\left(\frac{x}{a}\right)$

$\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$

CHAIN RULE

$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1^2 - (\cos x)^2}}\right)(-\sin x) = \frac{-\sin x}{\sqrt{1 - \cos^2 x}}$

But $\cos^2 x + \sin^2 x = 1$

$\Rightarrow 1 - \cos^2 x = \sin^2 x$

(see Log tables)

$= \frac{-\sin x}{\sqrt{\sin^2 x}} = \frac{-\sin x}{\sin x} = -1$

Q8 $y = \tan^{-1}\left(\frac{1}{x}\right)$ find $\frac{dy}{dx}_{(x=1)}$.

$y = \tan^{-1}\left(\frac{x}{a}\right)$

$\frac{dy}{dx} = \frac{a}{a^2+x^2}$

CHAIN RULE

$y = \tan^{-1}\left(\frac{(x^{-1})}{1}\right)$

$\frac{dy}{dx} = \left(\frac{1}{1^2 + \left(\frac{1}{x}\right)^2}\right)\left(-x^{-2}\right) = \frac{-1}{x^2(1+x^{-2})}$

$\frac{dy}{dx} = \frac{-1}{x^2+1}$ $\frac{dy}{dx}_{(x=1)} = \frac{-1}{(1)^2+1} = -\frac{1}{2}$

Q9 $y = \tan^{-1}(3x^2)$ $\frac{dy}{dx}_{(x=\frac{1}{3})} = ?$

$y = \tan^{-1}\left(\frac{x}{a}\right)$

$\frac{dy}{dx} = \frac{a}{a^2+x^2}$

CHAIN RULE

$y = \tan^{-1}\left(\frac{3x^2}{1}\right)$

$\frac{dy}{dx} = \left(\frac{1}{1^2 + (3x^2)^2}\right)(6x) = \frac{6x}{1+9x^4}$

$\frac{dy}{dx}_{(x=\frac{1}{3})} = \frac{6\left(\frac{1}{3}\right)}{1+9\left(\frac{1}{3}\right)^4} = \frac{2}{1+\frac{1}{9}} = \frac{2}{1}\cdot\frac{9}{10} = \frac{9}{5}$

Q10 $y = \tan^{-1} x$ show $\frac{d^2y}{dx^2}(1+x^2) + 2x\frac{dy}{dx} = 0$

$y = \tan^{-1}\left(\frac{x}{a}\right)$

$\frac{dy}{dx} = \frac{a}{a^2+x^2}$

$y = \tan^{-1}\left(\frac{x}{1}\right)$

$\frac{dy}{dx} = \frac{1}{1^2+x^2} = \frac{1}{1+x^2} = (1+x^2)^{-1}$

$\frac{d^2y}{dx^2} = -1(1+x^2)^{-2}(2x) = -2x(1+x^2)^{-2}$

$\frac{d^2y}{dx}(1+x^2) + 2x\frac{dy}{dx} = -2x(1+x^2)^{-2}(1+x^2) + 2x(1+x^2)^{-1}$

$= -2x(1+x)^{-1} + 2x(1+x^2)^{-1} = 0$