5th Year Maths

Christmas Exam Solutions

2012

Mr. Roche

1 (a) Simplify fully $\dfrac{x^2+4}{x^2-4} - \dfrac{x}{x+2}$

Write as single fraction

Lcm is x^2-4 as it is difference of 2 squares = $(x+2)(x-2)$

$$\frac{x^2+4}{(x+2)(x-2)} - \frac{x}{x+2}$$

$$= \frac{x^2+4-x(x-2)}{(x+2)(x-2)}$$

$$= \frac{\cancel{x^2}+4-\cancel{x^2}+2x}{(x+2)(x-2)}$$

Factorise HCF

$$= \frac{4+2x}{(x+2)(x-2)}$$

$$= \frac{2\cancel{(2+x)}}{\cancel{(x+2)}(x-2)}$$

$$= \frac{2}{x-2}$$

(b) Given that one of the Roots is an integer, solve the equation

$$6x^3 - 29x^2 + 36x - 9 = 0$$

use trial and error to find 1st root

If $f(k)=0$ then k is a Root. also $(x-k)$ is a factor

$f(3) = 6(3)^3 - 29(3)^2 + 36(3) - 9$
$= 162 - 270 + 108 - 9 = 0$

$\Rightarrow$ Integer Root is $x = 3$

$\Rightarrow$ $(x-3)$ is the related factor

Divide by $(x-3)$

$$\begin{array}{r l}
 & 6x^2 - 11x + 3 \\
x-3 \,\big)\! & \cancel{6x^3} - 29x^2 + 36x - 9 \\
 & \mp \cancel{6x^3} \pm 18x^2 \\
 & \quad -11\cancel{x^2} + 36x \\
 & \quad \pm \cancel{11x^2} \mp 33x \\
 & \qquad 3\cancel{x} - \cancel{9} \\
 & \qquad \mp \cancel{3x} \pm \cancel{9}
\end{array}$$

Factorise

$6x^2 - 11x + 3 = 0$
$(3x-1)(2x-3) = 0$
$\Rightarrow x = 1/3, \quad x = 3/2$

(c) Two of the Roots of the equation $ax^3 + bx^2 + cx + d = 0$ are p and $-p$. Show that $bc = ad$.

$f(p) = 0$

$f(-p) = 0$

$f(p) = ap^3 + bp^2 + cp + d = 0$ ①

$f(-p) = a(-p)^3 + b(-p)^2 + c(-p) + d = 0$
$-ap^3 + bp^2 - cp + d = 0$ ②

① + ②

$a\cancel{p^3} + bp^2 + c\cancel{p} + d = 0$
$-\cancel{a}p^3 + bp^2 - \cancel{c}p + d = 0$
$2bp^2 + 2d = 0$

divide by 2

$bp^2 + d = 0$
$p^2 = -d/b$ ③

① - ②

$ap^3 + b\cancel{p^2} + cp + \cancel{d} = 0$
$+ap^3 - \cancel{b}p^2 + cp - \cancel{d} = 0$
$2ap^3 + 2cp = 0$

divide by 2p

$ap^2 + c = 0$
$p^2 = -c/a$ ④

from ③ & ④

$\Rightarrow -d/b = -c/a$

multiply by -ba

$\Rightarrow ad = bc$

2 (a) Solve:

$$2x + 3y = 0 \quad (1)$$
$$x + y + z = 0 \quad (2)$$
$$3x + 2y - 4z = 9 \quad (3)$$

(3) + 4(2)

$$\begin{aligned} 3x + 2y - 4\cancel{z} &= 9 \\ 4x + 4y + 4\cancel{z} &= 0 \\ \hline 7x + 6y &= 9 \quad (4) \end{aligned}$$

(4) − 2(1)

$$\begin{aligned} 7x + \cancel{6y} &= 9 \\ -4x - \cancel{6y} &= 0 \\ \hline 3x &= 9 \\ x &= 3 \end{aligned}$$

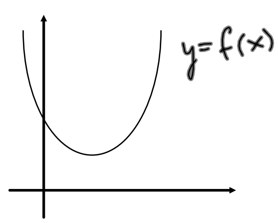
Sub into (1)

$$\begin{aligned} 2(3) + 3y &= 0 \\ 6 + 3y &= 0 \\ 3y &= -6 \\ y &= -2 \end{aligned}$$

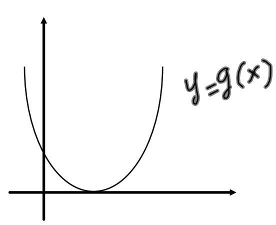
Sub into (2)

$$\begin{aligned} (3) + (-2) + z &= 0 \\ 1 + z &= 0 \\ z &= -1 \end{aligned}$$

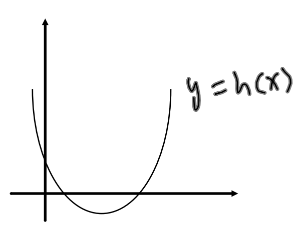
(b) (i) The graphs of three quadratic equations are shown in each case state the nature of the roots



2 imaginary Roots



2 Real Roots that are the same



2 Real Roots that are distinct

(ii) The equation $kx^2 + (1-k)x + k = 0$
has equal roots. Find the possible values of k.

equal roots ⇒

$\Delta = b^2 - 4ac = 0$

$a = k, \quad b = (1-k), \quad c = k$

$\Delta = (1-k)^2 - 4(k)(k) = 0$

Solve

$1 - 2k + k^2 - 4k^2 = 0$

$-3k^2 - 2k + 1 = 0$

$3k^2 + 2k - 1 = 0$

$(3k - 1)(k + 1) = 0$

$k = 1/3, \quad k = -1$

(c) One of the roots of $px^2 + qx + r = 0$ is n times the other root.
Express r in terms of p, q and n.

$x^2 - (R_1 + R_2)x + R_1R_2 = 0$

let one root $= k$ then the other $= nk$

Divide by p ⇒ $x^2 + \frac{q}{p}x + \frac{r}{p} = 0$

Sum of Roots $=$ $nk + k = k(n+1)$

$\frac{q}{p} = -k(n+1) \Rightarrow \frac{-q}{p(n+1)} = k$

Product of Roots $=$ $(nk)(k) = nk^2$

$\frac{r}{p} = nk^2$

express $r = \ldots$ $\Rightarrow r = pnk^2 = pn\left[\frac{-q}{p(n+1)}\right]^2 = pn\frac{q^2}{p^2(n+1)^2}$

$r = \frac{nq^2}{p(n+1)^2}$

Q.3 (a) Simplify fully $\frac{x+1}{x-1} - \frac{x-1}{x+1} - \frac{4}{x^2-1}$

Write as a single fraction

$x^2-1 = (x-1)(x+1)$

DIFFERENCE of 2 SQUARES

$\Rightarrow LCM = (x-1)(x+1)$

$$\frac{(x+1)(x+1) - (x-1)(x-1) - 4}{(x-1)(x+1)}$$

$$= \frac{x^2+2x+1 - [x^2-2x+1] - 4}{(x-1)(x+1)}$$

$$= \frac{\cancel{x^2} + 2x + \cancel{1} - \cancel{x^2} + 2x - \cancel{1} - 4}{(x-1)(x+1)}$$

Factorise numerator

$$= \frac{4x+4}{(x-1)(x+1)}$$

$$= \frac{4\cancel{(x+1)}}{(x-1)\cancel{(x+1)}}$$

$$= \frac{4}{x-1}$$

(b) (i) What is the Value of the jeweller's stock of old jewellery?

stock: 147 grams of 9-carat gold, 85 grams of 18-carat gold } Value = Value of gold content

Gold = €36 per gram

Purity 'carat rating' $C = \frac{24m_g}{m_t}$

m_g = mass of gold

m_t = total mass

$$C = \frac{24m_g}{m_t} \Rightarrow m_g = \frac{Cm_t}{24}$$

147 grams of 9-carat gold

$$m_g = \frac{9(147)}{24} = \frac{441}{8} \text{ grams}$$

85 grams of 18-carat gold

$$m_g = \frac{18(85)}{24} = \frac{255}{4} \text{ grams}$$

total gold content

$$\frac{441}{8} + \frac{255}{4} = \frac{951}{8} \text{ grams}$$

Total Value

$$36\left(\frac{951}{8}\right) = €4,279.50$$

(ii) The jeweller wants to make a 15-carat gold pennant weighing 21 grams.
He melts down some 9-carat gold and some 18-carat gold to do this.
How many grams of each should he use?

Let us use x grams of 9-carat gold and y grams of 18-carat gold

Total mass: $x + y = 21$ ①

Gold content mass: $m_g = \frac{c\,m_t}{24}$

9-carat + 18-carat = 15-carat: $\frac{9x}{24} + \frac{18y}{24} = \frac{(15)(21)}{24}$

multiply by $\frac{3}{8}$: $\Rightarrow x + 2y = 35$ ②

② - ①:

$$\begin{array}{r} x + 2y = 35 \\ -x - y = -21 \\ \hline y = 14 \end{array}$$

sub into ①: $x + 14 = 21 \Rightarrow x = 7$

(c) $(x-a)^2$ is a factor of $2x^3 - 5ax^2 + 8abx - 36a$, where $a \neq 0$.
Find the possible values of a and b,

expand: $(x-a)^2 = x^2 - 2ax + a^2$

divide
Remainder will be 0

$$\begin{array}{r|l} & 2x - a \\ x^2 - 2ax + a^2 & 2x^3 - 5ax^2 + 8abx - 36a \\ & \mp\cancel{2x^3} \pm 4ax^2 \mp 2a^2x \\ \hline & -a\cancel{x^2} + (8ab - 2a^2)x - 36a \\ & \pm\cancel{a}x^2 \mp 2a^2x \quad \pm a^3 \\ \hline & 0x + 0 \end{array}$$

Solve:

$\Rightarrow 8ab - 2a^2 - 2a^2 = 0$
$8ab - 4a^2 = 0$
$2b - a = 0$
$2b = a$
$b = a/2$

$-36a + a^3 = 0$
$a^2 = 36$
$a = \pm 6$

$b = -6/2 = -3$
$b = 6/2 = 3$

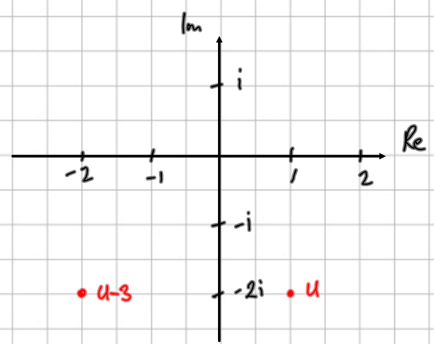
$\Rightarrow a = \pm 6, \; b = \pm 3$

4(a) Let $u = 1 + 2i$, where $i^2 = -1$

Plot on an Argand diagram (i) u (ii) $u - 3$

$u = 1 - 2i$

$u - 3 = 1 - 2i - 3 = -2 - 2i$



(b) Let $z = 2 + 3i$ (i) Find z^2 in the form $x + yi$, where $x, y \in \mathbb{R}$.

(ii) Show that $z^2 = 4z - 13$.

(iii) Show that $\bar{z}^2 + 13 = 4\bar{z}$ where $\bar{z}$ is the complex conjugate of z.

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expand (i)

$$z^2 = (2+3i)^2$$
$$= (2+3i)(2+3i)$$
$$= 4 + 6i + 6i + 9\cancel{i^2}$$
$$= -5 + 12i$$

expand (ii)

$$4z - 13 = 4(2+3i) - 13$$
$$= 8 + 12i - 13$$
$$= -5 + 12i$$
$$= z^2$$

(iii)

expand $\bar{z}^2 + 13$

$$\bar{z} = 2 - 3i$$
$$\bar{z}^2 = (2-3i)^2 = 4 - 12i + 9\cancel{i^2} = -5 - 12i$$
$$\bar{z}^2 + 13 = -5 - 12i + 13 = 8 - 12i$$

expand $4\bar{z}$

$$4\bar{z} = 4(2-3i) = 8 - 12i$$

$$\Rightarrow \bar{z}^2 + 13 = 4\bar{z}$$

(c) (i) Express $\frac{4+2i}{3-i}$ in the form $x+yi$, where $x, y \in \mathbb{R}$

multiply above and below by the conjugate of the denominator

notice difference of 2 Squares

$$= \frac{(4+2i)(3+i)}{(3-i)(3+i)}$$

$$= \frac{12+4i+6i \mp 2\cancel{i^2}}{9 \pm 1\cancel{i^2}}$$

$$= \frac{10+10i}{10}$$

$$= 1+i$$

(ii) Hence or otherwise, find the real numbers k and t such that

$$\underbrace{\left|\frac{4+2i}{3-i}\right|(k+5i)}_{LHS} = \underbrace{\frac{1}{\sqrt{2}}(7+(t-1)i)}_{RHS}$$

'Hence' means using the previous solution

$$\frac{4+2i}{3-i} = 1+i$$

modulus

$|a+bi| = \sqrt{a^2+b^2}$

$$\Rightarrow \left|\frac{4+2i}{3-i}\right| = |1+1i| = \sqrt{(1)^2+(1)^2} = \sqrt{2}$$

$$LHS = \sqrt{2}(k+5i) = \sqrt{2}k + 5\sqrt{2}i$$

$$RHS = \frac{1}{\sqrt{2}}(7+(t-1)i) = \frac{7}{\sqrt{2}} + \frac{(t-1)}{\sqrt{2}}i$$

Real = Real

Imaginary = Imaginary

$$\Rightarrow \sqrt{2}k = \frac{7}{\sqrt{2}}$$

$$k = \frac{7}{\sqrt{2}\sqrt{2}}$$

$$k = \frac{7}{2}$$

$$\Rightarrow 5\sqrt{2} = \frac{(t-1)}{\sqrt{2}}$$

$$5\sqrt{2}\sqrt{2} = t-1$$

$$10 = t-1$$

$$11 = t$$