PROJECT MATHS

Text & Tests

Leaving 3 Certificate

chapter 7

Complex Numbers

Test Yourself 7

Test yourself 7

1. (i) Write down one example of each of the following:

(a) A natural number that is greater than 25 and less than 40

(b) An integer which is less than -5 and a multiple of 2

(c) A rational number between 1 and 2

(d) An irrational number between 8 and 9.

(a) 26

(b) -6

(c) 1.5

(d) $\sqrt{70}$

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Test yourself 7

(ii) Given that $z_1 = 3 + 4i$ and $z_2 = -1 + 2i$,

(a) express $2z_1 - 3z_2$ in the form $a + bi$

(b) express $\frac{1}{z_1}$ in the form $a + bi$.

(c) If $z_1 + kz_2 = 6 - 2i$, find the value of k, where $k \in \mathrm{R}$.

(a) $2(3+4i) - 3(-1+2i)$

$6 + 8i + 3 - 6i$

$9 + 2i$

(b) $\frac{1}{(3+4i)} \cdot \frac{(3-4i)}{(3-4i)} = \frac{3-4i}{9+16} = \frac{3-4i}{25} = \frac{3}{25} - \frac{4}{25}i$

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Test yourself 7

(ii) Given that $z_1 = 3 + 4i$ and $z_2 = -1 + 2i$,

(a) express $2z_1 - 3z_2$ in the form $a + bi$

(b) express $\frac{1}{z_1}$ in the form $a + bi$.

(c) If $z_1 + kz_2 = 6 - 2i$, find the value of k, where $k \in \mathbb{R}$.

(c) $3+4i + k(-1+2i) = 6-2i$

$3+4i - k + 2ki = 6-2i$

$3-k=6$ | $4+2k=-2$

$-3=k$ | $2k=-6$

$k=-3$

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Find x and y if x and $y \in \mathbb{R}$

$x(3-2i) + y(-2+i) = 5-4i$

$3x - 2xi - 2y + yi = 5-4i$

$3x - 2y = 5$

$2(-2x + y = -4)$

$3x - 2y = 5$

$-4x + 2y = -8$

$+x = +3$

$-2(3) + y = -4$

$-6 + y = -4$

$y = 2$

Test yourself 7

2. (i) Simplify $3(1 + 5i) + i(3 - 2i)$ and express your answer in the form $p + qi$, where $p, q \in \mathbb{R}$.

$3 + 15i + 3i \pm 2i^2$

$5 + 18i$

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(ii) Let $z_1 = 10 - 2i$ and $z_2 = 2 - 3i$.

Show that $\frac{z_1}{z_2}$ can be written in the form $k(1 + i)$, where $k \in \mathbb{N}$ and hence write down the value of k.

$$\frac{(10-2i)(2+3i)}{(2-3i)(2+3i)} = \frac{20 + 30i - 4i \pm 6i^2}{4 \pm 9i^2}$$

$$= \frac{26 + 26i}{13} = 2 + 2i$$

$2(1+i) \Rightarrow k = 2$

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Test yourself 7

(iii) Write $\frac{3+2i}{2-i}$ in the form $x + yi, x, y \in \mathbb{R}$.

Hence find $|x + yi|$.

$$\frac{(3+2i)(2+i)}{(2+i)(2-i)} = \frac{6+3i+4i+2i^2}{4+1i^2}$$

$$= \frac{4+7i}{5} = \frac{4}{5} + \frac{7}{5}i$$

$$\text{Modulus} = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{7}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{49}{25}} = \sqrt{\frac{65}{25}} = \sqrt{\frac{13}{5}}$$

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Test yourself 7

3. (i) Write each of the following in the form $a + bi$:

(a) $3 + \sqrt{-16}$ (b) $2 - \sqrt{-9}$ (c) $\frac{-10 + \sqrt{-100}}{5}$

$3+4i$ $2-3i$ $= \frac{-10+10i}{5}$

$= -2+2i$

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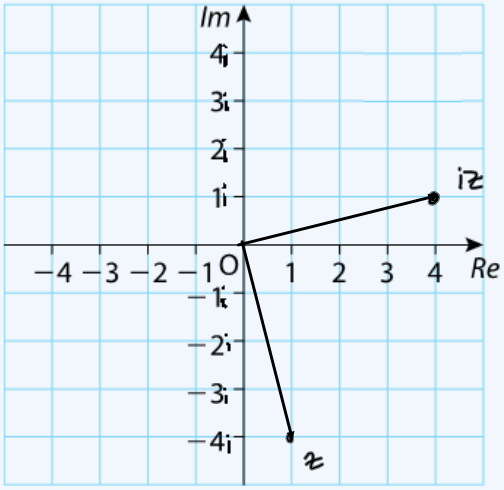
Test yourself 7

(ii) Let $z = 1 - 4i$, where $i^2 = -1$.
Plot z and iz on an Argand diagram.

$iz = i(1-4i) = i \pm 4i^2 = 4 + i$



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Test yourself 7

(iii) If $|2 + ki| = \sqrt{29}$, find the two values of $k \in \mathbb{R}$.

$\sqrt{2^2 + k^2} = \sqrt{29}$

$4 + k^2 = 29$

$k^2 = 25$

$k = \pm 5$

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Test yourself 7

4. (i) Write down an irrational number that has a value between 6 and 7.

$\sqrt{41}$

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Test yourself 7

(ii) Express each of the following in the form $a + bi$:

(a) $3(2 + 3i) - i(2 - 3i)$

(a) $6 + 9i - 2i + 3$

$3 + 7i$

(b) $\frac{2}{1 - 2i}$

(b) $\frac{2(1+2i)}{(1-2i)(1+2i)} = \frac{2+4i}{1+4}$

$= \frac{2}{5} + \frac{4}{5}i$

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Test yourself 7

(iii) If $z_1 = 3 - 4i$ and $z_2 = 2 + 3i$, express $z_1 z_2$ in the form $a + bi$.
Verify that $|z_1 z_2| = |z_1||z_2|$.

$(3-4i)(2+3i) = 6 + 9i - 8i \mp 12i^2 = 18 + i$

$|18+i| = \sqrt{18^2+1^2} = \sqrt{325} = 5\sqrt{13}$

$|z_1| = \sqrt{3^2+4^2} = \sqrt{25} = 5$

$|z_2| = \sqrt{2^2+3^2} = \sqrt{13}$

$5\sqrt{13} = (5)(\sqrt{13})$ QED

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Test yourself 7

5. (i) State whether each of the following is true or false:

(a) $-2 \in \mathbb{N}$ F (b) $4 \in \mathbb{Z}$ T (c) $\frac{2}{3} \in \mathbb{Q}$ T (d) $\sqrt{16}$ is irrational F

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Test yourself 7

(ii) Solve the equation $z^2 - 4z + 13 = 0$.
Express the roots in the form $a + bi$.

$a = 1$
$b = -4$
$c = 13$

$$z = \frac{+4 \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

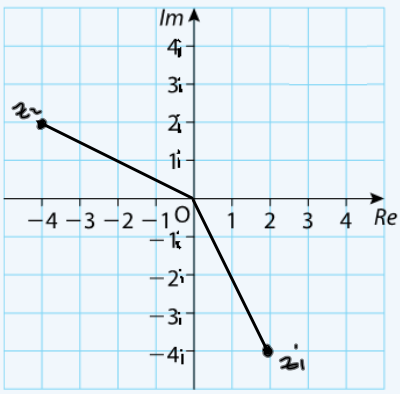
$$= \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

Test yourself 7

(iii) Plot $z_1 = 2 - 4i$ and $z_2 = -4 + 2i$ on an Argand diagram.
Find $|z_1|$ and $|z_2|$.
Write down two other complex numbers that have the same moduli as z_1.

$$|z_1| = \sqrt{2^2 + 4^2} = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5}$$

$$|z_2| = \sqrt{4^2 + 2^2} = 2\sqrt{5}$$



With same modulus of z_1 also

$z_3 = 2 + 4i$
$z_4 = -4 + 2i$

Test yourself 7

6. (i) Simplify $7(2 + i) + i(11 + 9i)$ and express your answer in the form $a + bi$, where $a, b \in \mathbb{R}$ and $i^2 = -1$.

$14 + 7i + 11i + 9i^2$

$5 + 18i$

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Test yourself 7

(ii) If $z_1 = 2 + 3i$ and $z_2 = 5 - i$, find

(a) $z_1 + z_2$ (b) $z_1 - z_2$.

Now investigate if $|z_1 + z_2| > |z_1 - z_2|$.

(a) $7 + 2i$

(b) $-3 + 4i$

$|7 + 2i| = \sqrt{7^2 + 2^2} = \sqrt{49 + 4} = \sqrt{53}$

$|-3 + 4i| = 5$

$\sqrt{53} > \sqrt{25}$

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Test yourself 7

(iii) Let $w = 3 - 4i$
Solve the equation $x + w = 3yi$, for $x, y \in \mathbb{R}$.

$x + 3 - 4i = 3yi$

$x + 3 = 0$ $\qquad -4i = 3yi$

$x = -3$ $\qquad \frac{-4}{3} = y$

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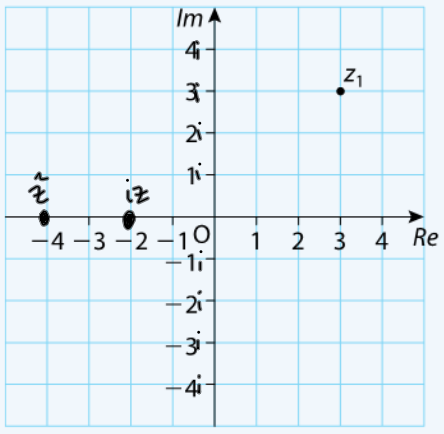
Test yourself 7

7. (i) If $z = 2i$, plot (a) z^2 and (b) iz on an Argand diagram.

$z^2 = (2i)^2 = +4i^2 = -4 + 0i$

$iz = -2i^2 = -2 + 0i$



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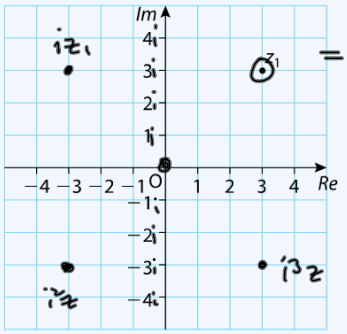
Test yourself 7

(ii) Write down, in the form $a + bi$, the complex number z_1 shown on the Argand diagram below.



$= 3 + 3i$

Now find the complex numbers iz_1, i^2z_1 and i^3z_1 and plot them on a copy of the Argand diagram above.

By examining the diagram, describe the geometrical transformation that has taken place.

$iz_1 = 3i + 3i^2 = -3 + 3i$

$-1(z_1) = -3 - 3i$

$-iz = 3 - 3i$

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Test yourself 7

(iii) If $z_1 = -4 + i$ is a root of the equation $z^2 + 8z + k = 0$, find the value of k and hence write down z_2, the other root of the equation.

Find z_1z_2 and hence find t, if $|z_1z_2| = t|z_1|$, for $t \in \mathbb{R}$.

$(-4+i)^2 + 8(-4+i) + k = 0$

$16 - 8i + i^2 - 32 + 8i + k = 0$

$-16 + k = 0$

$k = 16$

$z_2 = -4 - i$

$z_1z_2 = (-4-i)(-4+i)$

$= 16 \pm i^2$

$= 17$

$|z_1z_2| = |17| = 17$

$|z_1| = \sqrt{4^2 + 1^2} = \sqrt{17}$

$\Rightarrow 17 = t\sqrt{17}$

$\Rightarrow t = \sqrt{17}$

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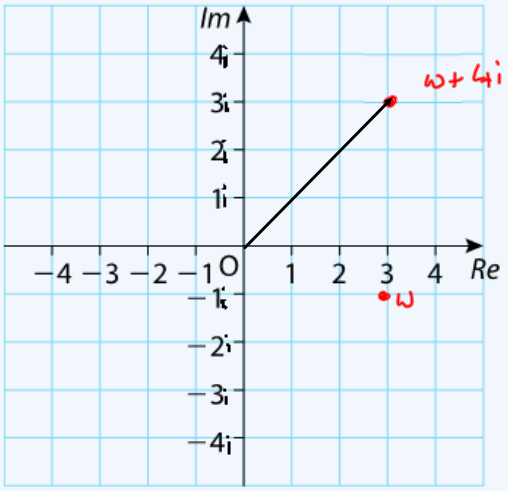
Test yourself 7

8. (i) If $w = 3 - i$, plot $w + 4i$ on an Argand diagram.
Now find $|w + 4i|$.



$w + 4i = 3 - i + 4i$
$= 3 + 3i$

$|3 + 3i| = \sqrt{3^2 + 3^2}$
$= \sqrt{9 + 9}$
$= \sqrt{18}$
$= 3\sqrt{2}$

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Test yourself 7

(ii) If $z = -2 + i$, express z^2 in the form $a + bi$, where $a, b \in \mathbb{R}$.
Hence solve the equation
$kz^2 = 2z + t$ for real k and real t.

$z^2 = (-2+i)^2 = +4 - 4i + i^2 = 3 - 4i$

$kz^2 = 2z + t$
$k(3 - 4i) = 2(-2 + i) + t$
$3k - 4ki = -4 + 2i + t$

Re=Re $\Rightarrow$ $3k = -4 + t$

Im=Im $\Rightarrow$ $-4k = 2$ $\quad k = -\frac{1}{2}$

$3\left(-\frac{1}{2}\right) = -4 + t$
$-\frac{3}{2} + 4 = t$
$-\frac{3}{2} + \frac{8}{2} = \frac{5}{2} = t$

$t = 2\frac{1}{2}$

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Test yourself 7

(iii) If $2 - i$ is a root of the equation $z^2 + pz + q = 0$, write down the other root. Hence find the value of p and the value of q.

other root is $2+i$

Sub in $(2-i)^2 + p(2-i) + q = 0$

$4 - 4i \pm 1i^2 + 2p - pi + q = 0$

$5 + 2p + q - 4i - pi = 0$

Re=Re $\Rightarrow$ $5 + 2p + q = 0$

Im=Im $\Rightarrow$ $-4 - p = 0$

$p = -4$

$\Rightarrow 5 + 2(-4) + q = 0$

$5 - 8 + q = 0$

$q = 3$

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Answers Test yourself 7

1. (ii) (a) $9 + 2i$ (b) $\frac{3}{25} - \frac{4}{25}i$ (c) $k = -3$
(iii) $x = 3, y = 2$

2. (i) $5 + 18i$ (ii) $k = 2$ (iii) $\frac{4}{5} + \frac{7}{5}i; \sqrt{\frac{13}{5}}$

3. (i) (a) $3 + 4i$ (b) $2 - 3i$ (c) $-2 + 2i$
(ii) $k = \pm 5$

4. (i) $2\sqrt{12}$ (ii) (a) $3 + 7i$ (b) $\frac{2}{5} + \frac{4}{5}i$
(iii) $18 + i$

5. (i) (a) F (b) T (c) T (d) F
(ii) $2 \pm 3i$
(iii) $|z_1| = 2\sqrt{5}, |z_2| = 2\sqrt{5}; -2 - 4i, 2 + 4i$

6. (i) $5 + 18i$
(ii) (a) $7 + 2i$ (b) $-3 + 4i$; Yes
(iii) $x = -3, y = -\frac{4}{3}$

7. (i) (a) $z^2 = -4 + 0i$ (b) $iz = -2 + 0i$
(ii) $3 + 3i; -3 + 3i; -3 - 3i; 3 - 3i;$
Rotation 360°
(iii) $k = 17; z_2 = -4 - i; t = \sqrt{17}$

8. (i) $3 + 3i; 3\sqrt{2}$
(ii) $3 - 4i; k = -\frac{1}{2}, t = \frac{5}{2}$
(iii) $2 + i; p = -4, q = 5$