Complex

We use the letter i to represent the imaginary number $\sqrt{-1}$

$i = \sqrt{-1}$
$\sqrt{-1}$ does not exist, no number squared gives – 1 that is why we say that it is imaginary.

$i^2 = \sqrt{-1}\sqrt{-1} = -1$
We always replace it with –1.

$i^3 = i^2 i = (-1)(i) = -i$
$i^4 = i^2 i^2 = (-1)(-1) = 1$

$\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2i$

Modulus
If $z = x + iy$
then the modulus $|z| = \sqrt{x^2 + y^2}$
The modulus is the distance from the complex number back to (0,0)

Quadratic Equation with Complex Roots
$z^2 + 2z + 2 = 0$

Use –b formula, will be negative number under root, turn into i's eg. $\sqrt{-4} = 2i$

Roots occur in conjugate pairs, z and $\bar{z}$
eg. $-1 + i$ & $-1 - i$

Complex Numbers Addition/ Subtraction
Add or subtract the real or i parts separately
$(a + bi) + (c + di) = (a + c) + (b + d)i$
$(a + bi) - (c + di) = (a - c) + (b - d)i$

Examples –
1. $5 + 2i + 3 + 6i = 8 + 8i$
2. $(3 - 3i) - (-2 - 5i) = 5 + 2i$

Multiplication
Just like Algebra BUT remember $i^2 = -1$

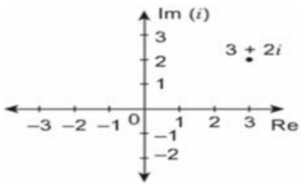
$(5 + 2i)(3 - 5i)$
$= 15 - 25i + 6i - 10i^2$
$= 15 - 19i - 10(-1)$
$= 15 - 19i + 10$
$= 25 - 19i$

Handy Rule
$(a + bi)(a - bi) = a^2 - b^2$ Test it

Complex Equations
Solve for x and y, both $\in R$
$x(3 + 4i) + y(2 - 3i) = 8 + 5i$

Remove brackets, let the real on the left equal the real on the right. Let the imaginary on the left equal the imaginary on the right. Solve resultant equations.

Argand Diagram
$3 + 2i$
$-2 + 3i$



Multiplying a complex number by i rotates it 90°

Conjugate
If $z = x + iy$
then the conjugate $\bar{z} = x - iy$

Division
To divide by a complex number we multiply above and below by the conjugate of the denominator.

$\dfrac{4 - 3i}{3 + 2i}$
$= \dfrac{4 - 3i}{3 + 2i} \cdot \dfrac{3 - 2i}{3 - 2i}$
$= \dfrac{12 - 8i - 9i + 6i^2}{9 - 6i + 6i - 4i^2}$
$= \dfrac{12 - 17i + 6(-1)}{9 - 4(-1)}$
$= \dfrac{12 - 17i - 6}{9 + 4}$
$= \dfrac{6 - 17i}{13}$
$= \dfrac{6}{13} - \dfrac{17}{13}i$

$x + iy$
Rectangular Form

$z = r(\cos\theta + i\sin\theta)$
Polar Form

Division/ Multiplication in Polar Form

$z_1 = 12(cos5\pi + isin5\pi)$
$z_2 = 3(\cos 2\pi + isin2\pi)$

$\dfrac{z_1}{z_2} = 4\big(\cos(5\pi - 2\pi) + isin(5\pi - 2\pi)\big)$
$= 4(\cos(3\pi + isin3\pi)$

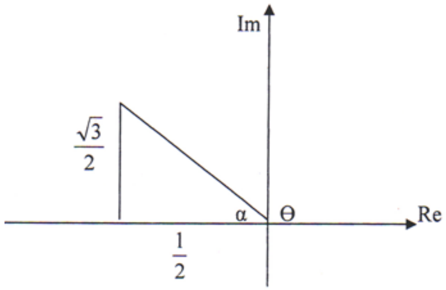
$z_1 z_2 = 36\big(\cos(5\pi + 2\pi) + isin(5\pi + 2\pi)\big)$
$= 36(\cos(7\pi + isin7\pi)$

Polar Form

$z = r(\cos\theta + i\sin\theta)$

$r = |z|$ and
θ is found drawing a sketch and using Tan ratio

Express $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ in the form $r(cos\theta + isin\theta)$



$$tan\alpha = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

$$\alpha = \frac{\pi}{3}$$

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$r = \left|-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right|$$

$$= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$$

Polar form of $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ is $1(cos\frac{2\pi}{3} + isin\frac{2\pi}{3})$

$= (cos\frac{2\pi}{3} + isin\frac{2\pi}{3})$

De Moivre's Theorem

If $z = \cos\theta + i\sin\theta$
$z^n = \cos n\theta + i\sin n\theta$

If $z = \cos\theta + i\sin\theta$
$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

Proof to be learned off.

Application of De Moivre's Theorem

$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$

Use De Moivre to write $(-2 - 2i)^5$ in the form $a + bi$

Write in polar form, use De Moivre, simplify result

General Polar Form to find Roots

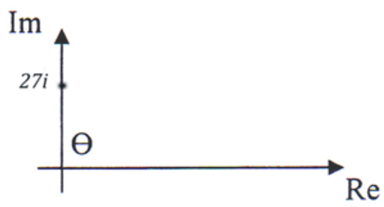
$rcos\theta + isin\theta = r\big(cos(\theta + 2\pi n) + isin(\theta + 2\pi n)\big)$

Steps

1. Write number in polar form
2. Write the number in General Polar Form
3. Apply De Moivre's Theorem
4. Let $n = 0,1,2,\ldots$.as required

Find Solutions to the equation $z^3 - 27i = 0$

$z^3 - 27i = 0$
$z^3 = 0 + 27i$
Write z^3 in polar form.



$\theta = \frac{\pi}{2}$ $\qquad r = |z| = 27$

$z^3 = rcos\theta + isin\theta$

$z^3 = 27\left(cos\frac{\pi}{2} + isin\frac{\pi}{2}\right)$

We now write z^3 in general polar form:
$rcos\theta + isin\theta = r\big(cos(\theta + 2\pi n) + isin(\theta + 2\pi n)\big)$

$$z^3 = 27\left(cos\left(\frac{\pi}{2} + 2\pi n\right) + isin\left(\frac{\pi}{2} + 2\pi n\right)\right)$$

Hence $z = 27^{\frac{1}{3}}\left(cos\left(\frac{\pi}{2} + 2\pi n\right) + isin\left(\frac{\pi}{2} + 2\pi n\right)\right)^{\frac{1}{3}}$

$$z = 3\left(cos\left(\frac{\pi + 4n\pi}{6}\right) + isin\left(\frac{\pi + 4n\pi}{6}\right)\right)$$

Let $n = 0$

$$z = 3\left(cos\frac{\pi}{6} + isin\frac{\pi}{6}\right) = 3\left(\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right) = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$$

Let $n = 1$

$$z = 3\left(cos\left(\frac{\pi + 4\pi}{6}\right) + isin\left(\frac{\pi + 4\pi}{6}\right)\right)$$

$$z = 3\left(-\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right)$$

$$z = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i$$

Let $n = 2$

$$z = 3\left(cos\left(\frac{\pi + 8\pi}{6}\right) + isin\left(\frac{\pi + 8\pi}{6}\right)\right)$$

$z = 3(0 + i(-1))$
$z = -3i$