# Sequences and Series

Sequences and series involve understanding idea of patterns and progression. You must be able to analyse and come up with a general term.

$\sum = sum$

$\sum_{r=1}^{4} T_r = T_1 + T_2 + T_3 + T_4$

**General Sequence Notation**

A sequence is a set of numbers or algebraic expressions defined by a rule.

**Example**

*The general term of a sequence is given by*
$U_n = 3n - 4$
*Find* $U_1, U_6, U_{n+1} - U_n$
*For what value of* $n$ *is* $U_n = 32$

$U_n = 3n - 4$

$U_1 = 3(1) - 4 = 3 - 4 = -1$

$U_6 = 3(6) - 4 = 18 - 4 = 14$

$U_{n+1} - U_n = 3(n+1) - 4 - (3n - 4)$
$= 3n + 3 - 4 - 3n + 4$
$= 3$

$U_n = 3n - 4$

$U_n = 32$
$3n - 4 = 32$
$3n = 36$
$n = 12$

**Limits**

The limit of a sequence is a unique number L such that $T_n$, the nth term of the sequence gets closer and closer to L for larger and larger values of $n$.

If a limit exists then the sequence is convergent. If it doesn't exist then the sequence is divergent.

$T_n \to L$ as $n \to \infty$ then $\lim_{n\to\infty} T_n = L$

$\lim_{n\to\infty} \frac{1}{n^p} = 0$ for $p > 0$

$$\lim_{n\to\infty} \frac{2n^2 + 15n - 1}{3n^2 - 3n + 2} = \frac{2}{3}$$

*Divide everything by the highest power of n*

$$\lim_{n\to\infty} \frac{\frac{2n^2}{n^2} + \frac{15n}{n^2} - \frac{1}{n^2}}{\frac{3n^2}{n^2} - \frac{3n}{n^2} + \frac{2}{n^2}}$$

$$\lim_{n\to\infty} \frac{2 + 0 - 0}{3 - 0 + 0}$$

$$= \frac{2}{3}$$

**Convergence**

A series is convergent if it has a limit
A series is divergent if it does not have a limit.

*State the values of x for which the series*
$\sum_{r=2}^{\infty} (4x - 1)^r$
*is convergent. Then find the sum to infinity in terms of x.*

$$\sum_{r=2}^{\infty} (4x-1)^r = (4x-1)^2 + (4x-1)^3 + (4x-1)^4 + \cdots$$

$$a = (4x-1)^2$$
$$r = 4x - 1$$

$S_\infty = \frac{a}{1-r}$ for $|r| < 1$ Sum to Infinity

Series convergent $|r| < 1$

$$|4x - 1| < 1$$
$$-1 < 4x - 1 < 1$$
$$0 < 4x < 2$$
$$0 < 4x < \frac{1}{2}$$

$$S_\infty = \frac{a}{1-r}$$
$$S_\infty = \frac{(4x-1)^2}{1-(4x-1)}$$
$$S_\infty = \frac{(4x-1)^2}{2-4x)}$$

## Arithmetic Sequences and Series

$2, 4, 6, 8, \ldots$

$a = first\ term$
$d = common\ difference$

$T_n = a + (n-1)d$ ***General Term***

$d = T_3 - T_2 = T_2 - T_1$

$S_n = \frac{n}{2}[2a + (n-1)d]$ ***Sum to n terms***

$T_n = S_n - S_{n-1}$

*Prove that the sequence* $T_n = 4n + 3$ *is arithmetic and find* $d$*, the common difference.*

$T_n = 4n + 3$
$T_{n+1} = 4(n+1) + 3$
$= 4n + 4 + 3$
$= 4n + 7$

$T_{n+1} - T_n = 4n + 7 - (4n + 3)$
$= 4$

*The first term of an arithmetic sequence is 7 and the common difference is* $-2$*. Which term of the sequence is* $-351$

$T_n = a + (n-1)d$
$= 7 + (n-1)(-2)$
$= 7 - 2n + 2$
$= 9 - 2n$
$9 - 2n = -351$
$360 = 2n$
$80 = n$

## Geometric Sequences and Series

$1, 4, 16, 64 \ldots$

$a = first\ term$
$r = common\ ratio$

$T_n = ar^{n-1}$ ***General Term***

$r = \frac{T_3}{T_2}$

$S_n = \frac{a(1-r^n)}{1-r}$ ***Sum to n terms***

$T_n = S_n - S_{n-1}$

$S_\infty = \frac{a}{1-r}$ for $|r| < 1$ ***Sum to Infinity***

*The first three terms of a geometric sequence is* $4, -12, 36$ *find* $a$ *the first term and* $r$*, the common ratio.*

$a = T_1 = 4$
$r = \frac{T_2}{T_1} = \frac{-12}{4} = -3$

*The first three terms of a geometric sequence are* $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ *Find the* $S_\infty$

$a = T_1 = \frac{1}{4}$

$r = \frac{T_2}{T_1} = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2}$

$S_\infty = \frac{\frac{1}{4}}{1 - \frac{1}{2}} = \frac{1}{2}$

## Reoccurring Sequence

*Express* $1.5\dot{4}\dot{7}$ *in the form* $\frac{p}{q}$

$1.5\dot{4}\dot{7} = 1.54747474747\ldots$

$= 1 + \frac{5}{10} + \frac{47}{1000} + \frac{47}{100000} + \cdots$

$= \frac{3}{2}\left(\frac{47}{10^3} + \frac{47}{10^5} + \cdots\right)$

$\frac{47}{10^3} + \frac{47}{10^5} + \cdots$ is an infinite geometric series

where $a = \frac{47}{100}$ and $r = \frac{1}{100}$

$1.5\dot{4}\dot{7} = \frac{3}{2} + \frac{a}{1-r}$

$1.5\dot{4}\dot{7} = \frac{3}{2} + \frac{\frac{47}{1000}}{1 - \frac{1}{100}}$

$1.5\dot{4}\dot{7} = \frac{3}{2} + \frac{\frac{47}{1000}}{\frac{99}{100}}$

$1.5\dot{4}\dot{7} = \frac{3}{2} + \frac{47}{990}$

$1.5\dot{4}\dot{7} = \frac{766}{495}$