

# Complex Numbers

## Revision (core)



15-January-2013

### Revision Exercise (Core)

1. Simplify  $\sqrt{80} - \sqrt{20}$ , expressing your answer in the form  $a\sqrt{b}$  where  $a, b \in \mathbb{N}$ .

$$= \sqrt{16 \times 5} - \sqrt{4 \times 5}$$

$$= 4\sqrt{5} - 2\sqrt{5}$$

$$= 2\sqrt{5}$$

Could use calculator!

3. Solve the equation  $z^2 + 4z + 3 = 0$ , giving your answer in the form  $a + bi$ .

quadratic formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 1 \quad b = 4 \quad c = 3$$

$$z = \frac{-4 \pm \sqrt{4^2 - 4(1)(3)}}{2(1)} = \frac{-4 \pm \sqrt{16 - 12}}{2}$$

$$= \frac{-4 \pm \sqrt{4}}{2} = \frac{-4 \pm 2}{2}$$

$$z = \frac{-2}{2} = -1 = 1 + 0i$$

$$\text{or } z = \frac{6}{2} = 3 = 3 + 0i \quad \checkmark$$

simple way

$$z^2 + 4z + 3 = 0$$

$$(z + 1)(z + 3) = 0$$

$$z = -1, z = -3$$



2.  $(x - 1) + yi = y + 4i$ ; find  $x$  and  $y$ .

$$\text{Re} = \text{Re}$$

$$x - 1 = y$$

$$\text{Im} = \text{Im}$$

$$y = 4$$

$$\Rightarrow x - 1 = 4$$

$$\Rightarrow x = 5$$

4. The roots of the quadratic equation  $z^2 + pz + q = 0$  are  $1+i$  and  $4+3i$ .  
Find the values of  $p$  and  $q$ .

$$\begin{aligned} z^2 - (\text{Sum Roots})z + (\text{Product Roots}) &= 0 \\ \text{Sum} &= (1+i) + (4+3i) = 5+4i \\ \Rightarrow -p &= 5+4i \\ p &= -5-4i \\ \text{Product} &= (1+i)(4+3i) = 4+3i+4i+3i^2 \\ &= 1+7i \\ \Rightarrow q &= 1+7i \end{aligned}$$

5. Let  $z$  be the complex number  $-1 + i\sqrt{3}$ .

- (i) Express  $z^2$  in the form  $a + bi$ .  
(ii) Find the value of the real number  $p$  such that  $z^2 + pz$  is real.

$$(i) (a+b)^2 = a^2 + 2ab + b^2 \quad (-1+i\sqrt{3})^2 = 1 - i2\sqrt{3} + 3i^2 = 2-2\sqrt{3}i$$

$$(ii) z^2 + pz = 2-2\sqrt{3}i + p(-1+i\sqrt{3}) \\ = 2-\underline{2\sqrt{3}i} - p + \underline{p\sqrt{3}i}$$

Real no.  $\Rightarrow$  Imaginary part = 0:

$$\Rightarrow -2\sqrt{3}i + p\sqrt{3}i = 0i$$

$$\Rightarrow p = 2$$

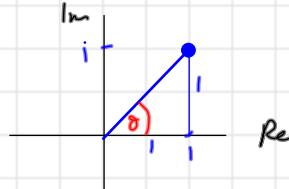
6. Express the complex number  $z = 1 + i$  in the form  $r(\cos \theta + i \sin \theta)$  and hence find a value for  $z^4$  in the form  $p + qi$  where  $p, q \in \mathbb{R}$ .

Polar Form

$$\begin{aligned} r &=? \\ \theta &=? \end{aligned}$$

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = ?$$



$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$x + iy = r(\cos \theta + i \sin \theta)$$

$$z = \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

$$z^4 = \left[ \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right]^4$$

de Moivre

$$= 4 \left( \cos 4\left(\frac{\pi}{4}\right) + i \sin 4\left(\frac{\pi}{4}\right) \right)$$

$$= 4 (\cos \pi + i \sin \pi)$$

$$= 4 (-1 + i0)$$

$$z^4 = -4$$

7. Express  $-1 + i\sqrt{3}$  in the form  $r(\cos \theta + i \sin \theta)$ .

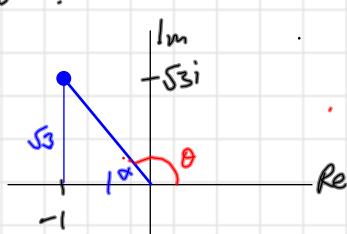
Polar Form

$$\begin{aligned} r &=? \\ \theta &=? \end{aligned}$$

$$\begin{aligned} \alpha & \\ \theta & \end{aligned}$$

$$r = \sqrt{1^2 + \sqrt{3}^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\theta = ?$$



$$\begin{aligned} \alpha &= \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) \quad \text{⑦} \\ \alpha &= \pi/3 \\ \theta &= \pi - \pi/3 \\ &= 2\pi/3 \end{aligned}$$

$$x + iy = r(\cos \theta + i \sin \theta)$$

$$-1 + i\sqrt{3} = 2 \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)$$

8. Show that  $2 + 3i$  is a root of  $z^2 - 4z + 13 = 0$ .  
Hence find the other root.

If  $k$  is a root  
 $f(k) = 0$

$$(2+3i)^2 - 4(2+3i) + 13$$

$$= 4 + 12i + 9i^2 - 8 - 12i + 13 \\ = 0 \quad \Rightarrow \text{it's a root}$$

Conjugate root theorem

$\Rightarrow$  other root is  $2 - 3i$

9. If  $z_1 = 2 + 3i$  and  $z_2 = 1 - 4i$ , investigate if  $|z_1| \cdot |z_2| = |z_1 \cdot z_2|$   
modulus

$$|a+bi| = \sqrt{a^2+b^2}$$

$$|z_1| = \sqrt{2^2+3^2} = \sqrt{4+9} = \sqrt{13}$$

$$|z_2| = \sqrt{1^2+4^2} = \sqrt{1+16} = \sqrt{17}$$

$$|z_1| \cdot |z_2| = (\sqrt{13})(\sqrt{17}) = \sqrt{221}$$

$$z_1 \cdot z_2 = (2+3i)(1-4i) = 2 - 8i + 3i + 12i^2 = 14 - 5i$$

$$|z_1 \cdot z_2| = \sqrt{14^2+5^2} = \sqrt{196+25} = \sqrt{221}$$

**10.** Write  $\frac{5 - 5i}{2 + i}$  in the form  $a + bi, a, b \in R$ .

multiply above and below  
by the conjugate of the denominator

difference of 2 squares

$$\begin{aligned}
 &= \frac{(5 - 5i)(2 - i)}{(2 + i)(2 - i)} \\
 &= \frac{10 - 5i - 10i + 5i^2}{4 + 1i^2} \\
 &= \frac{5 - 15i}{5} \\
 &= 1 - 3i
 \end{aligned}$$

$$i = i$$

$$i^2 = -1$$

$$i^3 = -i$$

$$\boxed{i^4 = 1}$$

Since 4 divides into 12  
 $\Rightarrow i^{12} = 1$

**11.** Simplify  $4i^{13} + 3i^3$ .

$$\begin{aligned}
 i^{13} &= i^{12} \cdot i = 1i \\
 i^3 &= -i
 \end{aligned}$$

$$\Rightarrow 4i^{13} + 3i^3 = 4i - 3i = 1i$$

12. Given that  $f(z)$  has roots  $z_1 = 2 + 3i$  and  $z_2 = -1 + 4i$ , find  $f(z)$ .

$$\text{Sum of Roots} = (2+3i) + (-1+4i) = 1+7i$$

$$\begin{aligned}\text{Product of Roots} &= (2+3i)(-1+4i) \\ &= -2+8i-3i+12i^2 \\ &= -14+5i\end{aligned}$$

$$z^2 - (1+7i)z - 14+5i = 0$$

13. Plot the complex numbers  $a = 3 + 3i$  and  $b = 1 - 2i$  on an Argand diagram. Plot the complex number  $a + b$  on the same diagram.

Find the complex number,  $c$ , that would translate

- (i)  $a$  to  $a + b$       (ii)  $b$  to  $a + b$       (iii)  $a$  to  $b$ .

$$a+b = (3+3i) + (1-2i) = 4+i$$

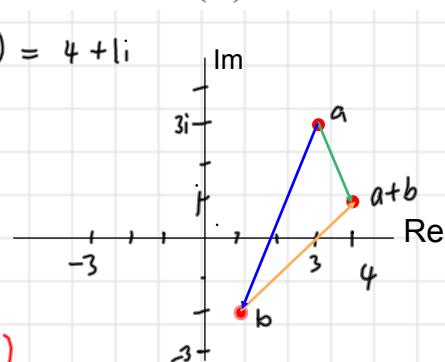
Translation  $\Rightarrow$  complex no. is added!

(i)

$$a \rightarrow a+b$$

$$\cancel{a+c} = \cancel{a} + b$$

$$\Rightarrow c = b = (1-2i)$$



(ii)

$$b \rightarrow a+b$$

$$\cancel{b+c} = a+\cancel{b}$$

$$\Rightarrow c = a = (3+3i)$$

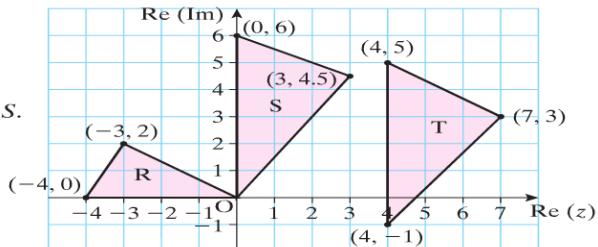
$a \rightarrow b$

(iii)

$$a+c = b$$

$$\Rightarrow c = b-a = (1-2i)-(3+3i) = (-2-5i)$$

14. In this diagram, describe the transformations needed for these:
- $R \rightarrow S$
  - $S \rightarrow T$
  - If  $z \in R$ , find  $z_1$  so that  $zz_1 \in S$ .
  - If  $z \in R$ , find  $z_3$  so that  $zz_1 + z_3 \in T$ .

(i)  $R \rightarrow S$ 

Rotation of  $-90^\circ$  (ie..  $90^\circ$  clockwise)  
stretched by  $\frac{3}{2}$

(ii)  $S \rightarrow T$ 

translated by  $(4-i)$