

**Revision Exercise (Extended-Response Questions)**

1. Given that  $p = 3\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$  and  $q = 2 - 2i\sqrt{3}$ .

- (i) Find  $pq$  in the form  $a + bi$ . (ii) Find  $|p|, |q|, |pq|, |p + q|$ .



(i)

$$p = 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \frac{3\sqrt{3}}{2} + \frac{3}{2}i \quad \checkmark$$

$$pq = \left(\frac{3\sqrt{3}}{2} + \frac{3}{2}i\right)(2 - 2\sqrt{3}i)$$

$$= \left(\frac{3\sqrt{3}}{2}\right)(2) - \left(\frac{3\sqrt{3}}{2}\right)(2\sqrt{3}i) + \left(\frac{3}{2}i\right)(2) + \left(\frac{3}{2}i\right)(2\sqrt{3}i) \quad \cancel{\times}$$

$$= 3\sqrt{3} - 9i + 3i + 3\sqrt{3} = 6\sqrt{3} - 6i \quad \checkmark$$

(ii)

*p's modulus given  
in polar form*

$$|a+bi| = \sqrt{a^2+b^2}$$

$$|p| = 3 \quad \checkmark$$

$$|q| = \sqrt{(2)^2 + (-2\sqrt{3})^2} = 4 \quad \checkmark$$

$$|pq| = \sqrt{(6\sqrt{3})^2 + (-6)^2} = 12 \quad \text{or } (|pq| = |p||q| = (3)(4) = 12)$$

$$p+q = \frac{3\sqrt{3}}{2} + \frac{3}{2}i + 2 - 2\sqrt{3}i = \left(\frac{3\sqrt{3}+4}{2}\right) + \left(\frac{3-4\sqrt{3}}{2}\right)i$$

use calculator

$$|p+q| = \sqrt{\left(\frac{3\sqrt{3}+4}{2}\right)^2 + \left(\frac{3-4\sqrt{3}}{2}\right)^2} = 5 \quad \checkmark$$

2. The complex number  $z = \frac{1+i\sqrt{3}}{1-i\sqrt{3}}$ .

- (i) Express  $z$  in the form  $a + bi$ . (ii) Plot  $z$  on an Argand diagram.

- (iii) Express  $z$  in the form of  $r(\cos \theta + i \sin \theta)$ . (iv) Show that  $z^3 = 1$ .

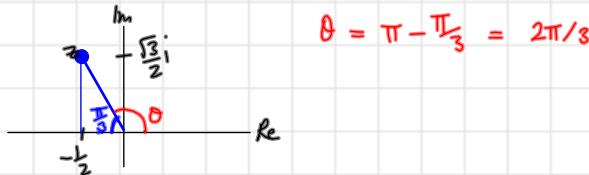
Multiply above  
and below by  
the conjugate

(i)

$$z = \frac{(1+\sqrt{3}i)(1+\sqrt{3}i)}{(1-\sqrt{3}i)(1+\sqrt{3}i)} = \frac{1+2\sqrt{3}i+3i^2}{1+3i^2} = \frac{4+2\sqrt{3}i}{4}$$

$$z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \quad \checkmark$$

(ii)



$$\theta = \pi - \frac{\pi}{3} = 2\pi/3$$

(iii)

$$r = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

$$z = 1 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$$

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(iv)

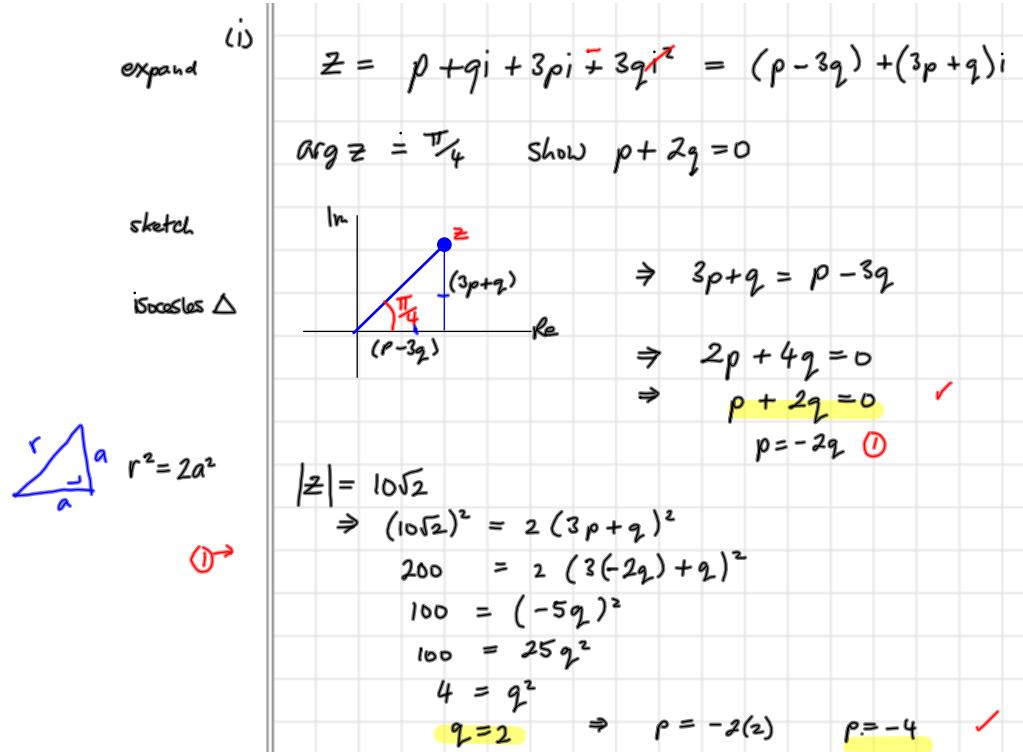
$$z^3 = \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)^3 = \cos 2\pi + i \sin 2\pi = 1 \quad \checkmark$$

3. The complex number  $z = (1 + 3i)(p + qi)$ , where  $p$  and  $q \in \mathbb{R}$  and  $p > 0$ .

(i) Write  $z$  in the form  $a + bi$ .

Given that the argument of  $z = \frac{\pi}{4}$ , show that  $p + 2q = 0$ .

(ii) Given also that  $|z| = 10\sqrt{2}$ , find the values of  $p$  and  $q$ .



4.  $z_1 = 3\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$  and  $z_2 = \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$ ; find

(i) $ z_1 z_2 $	(ii) $\arg(z_1 z_2)$	(iii) $ z_1 ^2$
(iv) $ z_2 ^2$	(v) $\arg(z_1^2)$	(vi) $\arg(z_2^2)$

(vii) Determine if each of the following statements is true or false for any two complex numbers  $z, w \in \mathbb{C}$ .

(a)  $|zw| = |z||w|$       (b)  $\arg(zw) = \arg(z) + \arg(w)$

	(i) $ z_1 z_2  =  z_1  \times  z_2  = (3)(1) = 3$
	(ii) $\arg(z_1 z_2) = \arg z_1 + \arg z_2 = \frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12}$
	(iii) $ z_1 ^2 = (3)^2 = 9$
	(iv) $ z_2 ^2 = (1)^2 = 1$
de Moivre	(v) $\arg(z_1^2) = 2(\arg z_1) = \frac{\pi}{3}$
	(vi) $\arg(z_2^2) = 2(\arg z_2) = \frac{\pi}{2}$
	(vii) Both statements are true

5. If  $z = 2\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$ , find  $z^2, z^4$  and  $z^6$ .

(i) Plot  $z^2, z^4, z^6$ .

(ii) Describe the transformation that occurs as  $z^2$  is multiplied each time.

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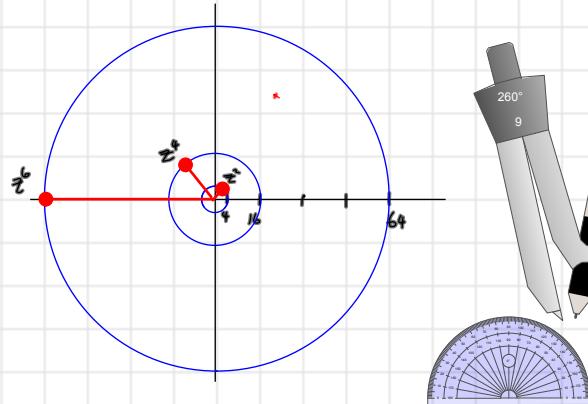
$$z^4 = z^2 \cdot z^2$$

$$z^6 = z^2 \cdot z^2 \cdot z^2$$

$$z^2 = 2^2 \left(\cos \frac{2\pi}{6} + i \sin \frac{2\pi}{6}\right) = 4 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$$

$$z^4 = 2^4 \left(\cos \frac{4\pi}{6} + i \sin \frac{4\pi}{6}\right) = 16 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$$

$$z^6 = 2^6 \left(\cos \frac{6\pi}{6} + i \sin \frac{6\pi}{6}\right) = 64 \left(\cos \pi + i \sin \pi\right)$$



Describe:

each time there is a rotation anticlockwise of  $\frac{\pi}{3}$   
and stretching ( $\times 4$ )

✓

6. Express  $\frac{\sqrt{3} + i}{1 + i\sqrt{3}}$  in the form  $r(\cos \theta + i \sin \theta)$ . Hence evaluate  $\left(\frac{\sqrt{3} + i}{1 + i\sqrt{3}}\right)^6$

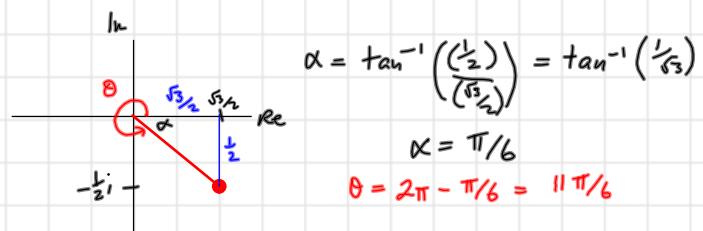
$$z = \frac{(\sqrt{3} + i)(1 - \sqrt{3}i)}{(1 + \sqrt{3}i)(1 - \sqrt{3}i)} = \frac{\sqrt{3} - 3i + i - \sqrt{3}i}{1 - 3i^2}$$

$$= \frac{2\sqrt{3} - 2i}{4} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$$

Polar form

$\theta = ?$

$r = ?$



$$r = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{1} = 1$$

$$z = 1 \left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)$$

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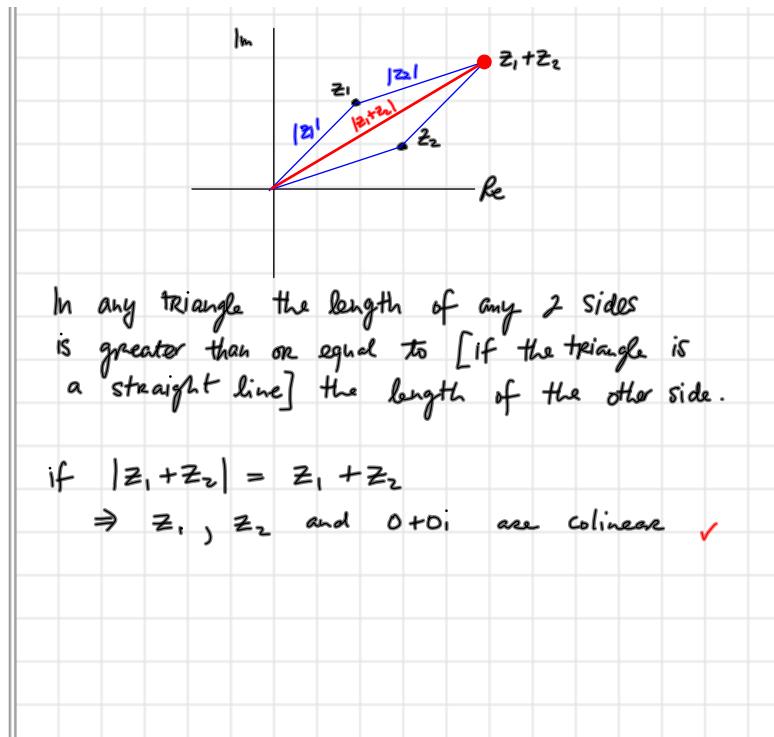
$$z^6 = \cos 6\left(\frac{11\pi}{6}\right) + i \sin 6\left(\frac{11\pi}{6}\right) = \cos 11\pi + i \sin 11\pi = -1$$

7. Plot any two complex numbers  $z_1, z_2$ .

By completing the parallelogram, find the complex number  $z_1 + z_2$ .

Using this parallelogram, describe geometrically a proof for the triangle inequality  $|z_1 + z_2| \leq |z_1| + |z_2|$ .

Under what conditions is  $|z_1 + z_2| = |z_1| + |z_2|$ ?



In any triangle the length of any 2 sides  
is greater than or equal to [if the triangle is  
a straight line] the length of the other side.

if  $|z_1 + z_2| = z_1 + z_2$   
 $\Rightarrow z_1, z_2$  and  $0+0i$  are collinear ✓

8.  $z$  is said to be the reciprocal of  $w$  if  $zw = 1$ .

- By letting  $z = a + bi$  and  $w = c + di$ , find two algebraic relationships between the real parts and imaginary parts  $a, b, c$  and  $d$ .
- Using simultaneous equations, find  $a$  and  $b$  in terms of  $c$  and  $d$ .
- Prove that  $\frac{1}{z} = \frac{z}{|z|^2}$ . (iv) Plot  $z, \frac{1}{z}$  and  $\bar{z}$  on the same Argand diagram.
- Prove that  $\frac{1}{z}$  and  $z$  are always collinear with  $0 + 0i$ .

$(i) \quad z = a + bi \quad w = c + di$ $zw = 1 \quad \Rightarrow \quad (a+bi)(c+di) = 1$ $ac + adi + bci + bdi^2 = 1$ $(ac - bd) + (ad + bc)i = 1 + 0i$ $\Rightarrow \quad ac - bd = 1 \quad \textcircled{1} \quad \text{and} \quad ad + bc = 0 \quad \textcircled{2}$	$a = -bc/d$ $(-bc/d)c - bd = 1$ $-b(c^2/d + d) = 1$ $b\left(\frac{c^2 + d^2}{d}\right) = -1$ $b = \frac{-d}{c^2 + d^2}$	$b = -ad/c$ $ac - (-ad/c)d = 1$ $ac + ad^2/c = 1$ $a\left(\frac{c^2 + d^2}{c}\right) = 1$ $a = \frac{c}{c^2 + d^2}$
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from  $\textcircled{2}$  (ii)  
sub into  $\textcircled{1}$   
write single fraction ✓

8.  $z$  is said to be the reciprocal of  $w$  if  $zw = 1$ .

- By letting  $z = a + bi$  and  $w = c + di$ , find two algebraic relationships between the real parts and imaginary parts  $a, b, c$  and  $d$ .
- Using simultaneous equations, find  $a$  and  $b$  in terms of  $c$  and  $d$ .
- Prove that  $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ .
- Plot  $z, \frac{1}{z}$  and  $\bar{z}$  on the same Argand diagram.
- Prove that  $\frac{1}{z}$  and  $z$  are always collinear with  $0 + 0i$ .

(iii)

$$\text{Prove } \frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

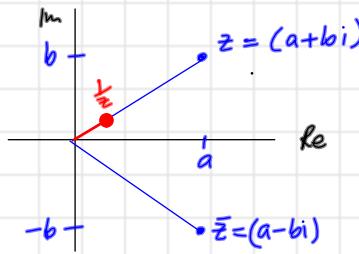
$$\frac{1}{z} = \frac{1(a-bi)}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2} = \frac{a+bi}{a^2+b^2}$$

$$|z| = \sqrt{a^2+b^2}$$

$$\frac{\bar{z}}{|z|^2} = \frac{a+bi}{(\sqrt{a^2+b^2})^2} = \frac{a+bi}{a^2+b^2}$$

(iv)

Plot  $z, \frac{1}{z}$  and  $\bar{z}$



Prove  $z, \frac{1}{z}$  and  $0+0i$  are collinear.

$$\text{Since } \frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

$$\text{if } z = r(\cos\theta + i\sin\theta) \\ \Rightarrow \frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta)$$

These have the same argument  
 $\Rightarrow$  they're collinear.