Extended Complex Number Questions 9-11 — January 28, 2013

9. Prove that
(i) the conjugate of a sum of complex numbers is equal to the sum of the conjugates
(ii) the conjugate of the difference of complex numbers is equal to the difference of the conjugates
(iii) the conjugate of a quotient of complex numbers is equal to the quotient of the conjugates.
(iv) the conjugate of a product of complex numbers is equal to the product of the conjugates.

let $z_1 = a+bi$
$z_2 = c+di$

$\Rightarrow \bar{z}_1 = a-bi$
$\bar{z}_2 = c-di$

(i) to show $\overline{(z_1+z_2)} = \bar{z}_1+\bar{z}_2$

$z_1+z_2 = a+bi+c+di = (a+c)+(b+d)i$

$\overline{z_1+z_2} = (a+c)-(b+d)i$

$\bar{z}+\bar{z} = a-bi+c-di = (a+c)-(b+d)i$ ✓

(ii) to show $\overline{z_1-z_2} = \bar{z}_1-\bar{z}_2$

$z_1-z_2 = a+bi-(c+di) = (a-c)+(b-d)i$

$\overline{(z_1-z_2)} = (a-c)-(b-d)i$

$\bar{z}_1-\bar{z}_2 = a-bi-(c-di) = a-c-bi+di$

$= (a-c)+(b-d)i$ ✓

quotient = divide

(iii) to show $\overline{(z_1/z_2)} = \bar{z}_1/\bar{z}_2$

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac+cdi+bci\overset{+}{-}bd\cancel{i^2}}{c^2\overset{+}{-}d^2\cancel{i^2}}$$

$$= \frac{(ac+bd)+(bc-cd)i}{c^2+d^2}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{(ac+bd)-(bc-cd)i}{c^2+d^2}$$ ✓

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{(a-bi)(c+di)}{(c-di)(c+di)} = \frac{ac+adi-bci\overset{+}{-}bd\cancel{i^2}}{c^2\overset{+}{-}d^2\cancel{i^2}}$$

$$= \frac{(ac+bd)-(bc-ad)i}{c^2+d^2}$$ ✓

Product = multiply

(iv) to show $\overline{z_1 z_2} = \bar{z}_1 . z_2$

$z_1 . z_2 = (a+bi)(c+di) = ac+adi+bci\overset{-}{+}bd\cancel{i^2}$

$= (ac-bd)+(ad+bc)i$

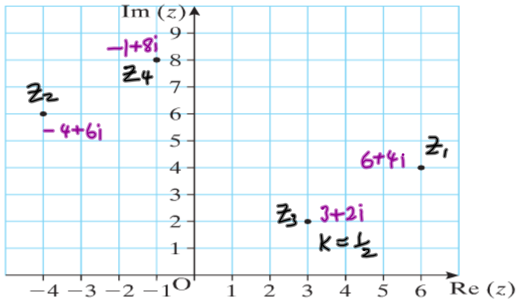
$\overline{z_1 . z_2} = (ac-bd)-(ad+bc)i$ ✓

$\bar{z}_1 . \bar{z}_2 = (a-bi)(c-di) = ac-adi-bci\overset{-}{+}bd\cancel{i^2}$

$= (ac-bd)-(ad+bc)i$ ✓

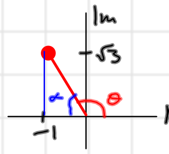
10. (a) Given $w = -1 + \sqrt{3}i$, where $i^2 = -1$.
(i) Write w in polar form.
(ii) Use de Moivre's theorem to solve $z^2 = -1 + \sqrt{3}i$, giving your answer in rectangular form.

(b) Four complex numbers z_1, z_2, z_3 and z_4 are shown on an Argand diagram. They satisfy the following conditions:
$z_2 = i\,z_1$
$z_3 = k\,z_1$, where $k \in R$
$z_4 = z_2 + z_3$.
(Note: the same scale was used on both axes.)
(i) Identify which number is which by labelling each point in the diagram.
(ii) Write down an appropriate value of k.
(iii) State which condition helped to identify the numbers first. Explain your answer. (Adapted from SEC *project maths* paper 1, 2011)



a (i)

Polar



$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

$\alpha = \tan^{-1}(\sqrt{3}/1) = \pi/3$

$\theta = \pi - \pi/3 = 2\pi/3$

$w = 2(\cos 2\pi/3 + i \sin 2\pi/3)$

de Moivre (ii)

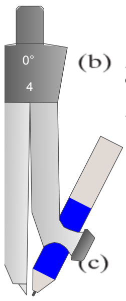
$z^2 = w^2 = 2^2(\cos 2(2\pi/3) + i \sin 2(2\pi/3))$

$= 4(\cos 4\pi/3 + i \sin 4\pi/3)$

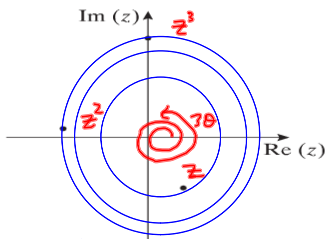
Calculator

$= 4(-1/2 - \sqrt{3}/2) = -2 - 2\sqrt{3}$

11. (a) (i) Write the complex number $1 - i$ in polar form.
(ii) Use de Moivre's theorem to evaluate $(1-i)^9$, giving your answer in rectangular form.

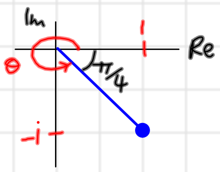


(b) A complex number z has a modulus greater than 1. The three numbers z, z^2 and z^3 are shown on an Argand diagram. One of them lies on the imaginary axis, as shown.
(i) Label the points to show which point corresponds to which number.
(ii) Find θ, the argument of z. $3\theta = 2\frac{1}{4}\pi = \frac{9}{4}\pi \Rightarrow \theta = \frac{3\pi}{4}$
(iii) Explain the significance of knowing that the modulus of z is greater than 1.
$\rightarrow |z| < |z^2| < |z^3|$



(c) Consider the complex number $z = a + ai$, $a > 1$, $a \in R$.
(i) Find the complex numbers z^2, z^4, z^6, etc., giving your answers in rectangular form.
(ii) Describe geometrically the pattern formed by z, z^2, z^4, z^6, ...
(Adapted from SEC *project maths* paper 1, 2010)

(i) $z = 1 - i$



$r = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta = -\pi/4$

OR $\theta = 2\pi - \pi/4$

$\theta = 7\pi/4$

$z = \sqrt{2}(\cos 7\pi/4 + i \sin 7\pi/4)$

de moivre (ii)

$z^9 = (\sqrt{2})^9(\cos 9(7\pi/4) + i \sin 9(7\pi/4))$

$= 16\sqrt{2}(\cos 63\pi/4 + i \sin 63\pi/4)$

(c) $z = a + ai \qquad a > 1, \; a \in \mathbb{R}$

(i)

$z = a + ai = a(1+i)$

$z^2 = a^2(1+i)^2 = a^2(1 + 2i \mp 1\cancel{i^2}) = 2a^2 i$

$z^4 = (2a^2 i)^2 = -4a^4$

$z^6 = (-4a^4)(2a^2 i) = -8a^6 i$

$z^8 = (-8a^6 i)(2a^2 i) = \pm 16 a^8 \cancel{i^2} = 16a^8$

(ii)

Don't worry as we haven't covered geometric patterns yet! ⇒

The next term is generated by multiplying by $2a^2 i$ ⇒ geometric pattern