Advanced Revision Questions

Complex Numbers

Revision Exercise (Advanced)

1. If $z = x + iy$ and $3(z - 1) = i(z + 1)$, find the value of x and the value of y.

Sub in $x+iy$ for z: $3(x + iy - 1) = i(x + iy + 1)$

expand: $3x + 3yi - 3 = ix \mp y\cancel{i^2} + 1i$

Re = Re: $3x - 3 = -y$

$3x + y = 3$ ①

Im = Im: $3y = x + 1$

$-x + 3y = 1$ ②

Solve

① + 3②

$$\begin{aligned} 3x + y &= 3 \\ -3x + 9y &= 3 \\ \hline 10y &= 6 \\ y &= 6/10 \\ y &= 3/5 \end{aligned}$$

Sub in:

$$\begin{aligned} 3y &= x + 1 \\ 3\left(\tfrac{3}{5}\right) - 1 &= x \\ \tfrac{9}{5} - 1 &= x \\ x &= 4/5 \end{aligned}$$

2. Given that $2 + 3i$ is a root of $2z^3 - 9z^2 + 30z - 13 = 0$, find the other two roots.

Conjugate Root theorem

If $2+3i$ is a root so is $2-3i$

Form quadratic

Sum of Roots: $(2+3i)+(2-3i) = 4+0i$

Product of Roots $= (2+3i)(2-3i) = 4 \pm 9i^{\cancel{2}} = 13$

$z^2-(R_1+R_2)z+(R_1R_2)=0$

$\Rightarrow \quad z^2-4z+13=0$

Divide to get 3rd linear factor

$$
\begin{array}{r|l}
 & 2z - 1 \\
z^2-4z+13 & 2\cancel{z^3} - 9z^2 + 30z - 13 \\
 & \mp \cancel{2z^3} \pm 8z^2 \mp 26z \\
 & -\cancel{z^2} + 4\cancel{z} - \cancel{13} \\
 & \pm \cancel{z^2} \mp \cancel{4z} \pm \cancel{13}
\end{array}
$$

Use factor to write root

factor: $(2z-1)=0$

$2z = 1$

$z = 1/2$

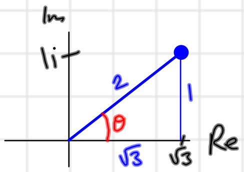
3. Express $\sqrt{3} + i$ in the form $r(\cos\theta + i\sin\theta)$.
Use de Moivre's theorem to simplify $(\sqrt{3} + i)^{11}$.

Polar form

$r=?$

$\theta=?$

$r = \sqrt{\sqrt{3}^2+1^2} = \sqrt{3+1} = \sqrt{4} = 2$



$\theta = \tan^{-1}(1/\sqrt{3}) = \pi/6$

$\sqrt{3}+i = 2(\cos \pi/6 + i\sin \pi/6)$

$(\sqrt{3}+i) = [2(\cos \pi/6 + i\sin \pi/6]$

de Moivre

$= 2^{11}[\cos 11(\pi/6) + i\sin 11(\pi/6)]$

$= 2^{11}(\sqrt{3}/2 - \frac{1}{2}i)$

$= 2^{10}(\sqrt{3} - 1i)$

$= 1024\sqrt{3} - 1024\,i$

4. The roots of the quadratic equation $z^2 + pz + q = 0$ are $1 + i$ and $4 + 3i$. Find the values of p and q.

$$z^2 - (R_1 + R_2)z + (R_1R_2) = 0$$

$$\Rightarrow \quad p = -(R_1 + R_2) \quad \text{and} \quad q = R_1R_2$$

Sum of Roots

$$(1+i)+(4+3i) = 5 + 4i$$

$$p = -5-4i$$

Product of Roots

$$(1+i)(4+3i) = 4 + 3i + 4i \mp 3\cancel{i^2} = 1 + 7i$$

$$q = 1 + 7i$$

Other method

Sub. in Roots

$f(K) = 0$

and solve

Sim. equations

5. If $w_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $w_2 = (w_1)^2$, find w_2.
Prove that $w_1 + w_2 = -1$.

$(a+b)^2 = a^2 + 2ab + b^2$

$$w_1^2 = \left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right)^2 = \tfrac{1}{4} - 2\left(\tfrac{\sqrt{3}}{4}i\right) \mp \tfrac{3}{4}\cancel{i^2}$$

$$\Rightarrow \quad w_2 = -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i$$

$$w_1 + w_2 = \left(-\tfrac{1}{2} + \cancel{\tfrac{\sqrt{3}}{2}i}\right) + \left(-\tfrac{1}{2} - \cancel{\tfrac{\sqrt{3}}{2}i}\right)$$

$$= -1$$

6. If $p = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, find $\overline{p}$, the complex conjugate of p. Prove that $p\overline{p}$ is a real number.

use calculator

to write in Cartesian form

$$p = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 1 + \sqrt{3}\,i$$

$$\bar{p} = 1 - \sqrt{3}\,i$$

$$p\bar{p} = (1 + \sqrt{3}\,i)(1 - \sqrt{3}\,i)$$

difference of 2 squares

$$= 1 \pm 3i^{\cancel{2}}$$

$$= 4$$