

## Advanced Revision Questions

## Complex Numbers

**Revision Exercise (Advanced)**

1. If  $z = x + iy$  and  $3(z - 1) = i(z + 1)$ , find the value of  $x$  and the value of  $y$ .

Sub in  $x+iy$  for  $z$

$$3(x+iy-1) = i(x+iy+1)$$

expand

$$3x + 3yi - 3 = ix + iy + 1$$

$Re = Re$

$$3x - 3 = -y$$

$$3x + y = 3 \quad \textcircled{1}$$

$Im = Im$

$$3y = x + 1$$

$$-x + 3y = 1 \quad \textcircled{2}$$

Solve

$$\textcircled{1} + 3\textcircled{2}$$

$$3x + y = 3$$

$$-3x + 9y = 3$$

$$10y = 6$$

$$y = 6/10$$

$$y = 3/5$$

Sub in:

$$3y = x + 1$$

$$3(\frac{3}{5}) - 1 = x$$

$$\frac{9}{5} - 1 = x$$

$$X = 4/5$$

2. Given that  $2 + 3i$  is a root of  $2z^3 - 9z^2 + 30z - 13 = 0$ , find the other two roots.

Conjugate Root theorem

If  $2+3i$  is a root so is  $2-3i$

Form quadratic

$$\text{Sum of Roots} = (2+3i) + (2-3i) = 4+0i$$

$$\text{Product of Roots} = (2+3i)(2-3i) = 4+9i^2 = 13$$

$$z^2 - (R_1+R_2)z + (R_1R_2) = 0$$

$$\Rightarrow z^2 - 4z + 13 = 0$$

Divide to get 3rd linear factor

$$\begin{array}{r} & \underline{2z-1} \\ z^2 - 4z + 13 & ) \underline{2z^3 - 9z^2 + 30z - 13} \\ & \underline{+ 2z^3 - 8z^2 + 26z} \\ & \underline{-z^2 + 4z - 13} \\ & \underline{+z^2 - 4z + 13} \end{array}$$

Use factor to write root

$$\text{factor: } (2z-1) = 0$$

$$2z = 1$$

$$z = 1/2$$

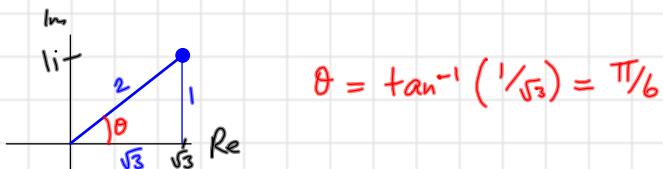
3. Express  $\sqrt{3} + i$  in the form  $r(\cos \theta + i \sin \theta)$ .  
Use de Moivre's theorem to simplify  $(\sqrt{3} + i)^{11}$ .

Polar form

$$r=?$$

$$\theta=?$$

$$r = \sqrt{\sqrt{3}^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$$



$$\theta = \tan^{-1}(1/\sqrt{3}) = \pi/6$$

$$\sqrt{3} + i = 2(\cos \pi/6 + i \sin \pi/6)$$

$$(\sqrt{3} + i)^{11} = [2(\cos \pi/6 + i \sin \pi/6)]^{11}$$

de Moivre

$$= 2^{11} [\cos 11(\pi/6) + i \sin 11(\pi/6)]$$

$$= 2^{11} (\frac{\sqrt{3}}{2} - \frac{1}{2}i)$$

$$= 2^{10} (\sqrt{3} - 1i)$$

$$= 1024\sqrt{3} - 1024i$$

4. The roots of the quadratic equation  $z^2 + pz + q = 0$  are  $1+i$  and  $4+3i$ .  
Find the values of  $p$  and  $q$ .

$$z^2 - (R_1 + R_2)z + (R_1 R_2) = 0$$

$$\Rightarrow p = -(R_1 + R_2) \quad \text{and} \quad q = R_1 R_2$$

Sum of Roots

$$(1+i) + (4+3i) = 5+4i$$

$$p = -5-4i$$

Product of Roots

$$(1+i)(4+3i) = 4+3i+4i-3 = 1+7i$$

$$q = 1+7i$$

Other method

Sub. in Roots

$f(k)=0$   
and solve  
sim. equations

5. If  $w_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$  and  $w_2 = (w_1)^2$ , find  $w_2$ .  
Prove that  $w_1 + w_2 = -1$ .

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$w_1^2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2 = \frac{1}{4} - 2\left(\frac{\sqrt{3}}{4}i\right) - \frac{3}{4}i^2$$

$$\Rightarrow w_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$w_1 + w_2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$= -1 \quad \text{Smiley Face}$$

6. If  $p = 2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$ , find  $\bar{p}$ , the complex conjugate of  $p$ . Prove that  $p\bar{p}$  is a real number.

use calculator  
to write in Cartesian form

$$p = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 1 + \sqrt{3}i$$

$$\bar{p} = 1 - \sqrt{3}i$$

$$p\bar{p} = (1 + \sqrt{3}i)(1 - \sqrt{3}i)$$

$$= 1 + 3i^2$$

$$= 4 \quad \text{Smiley Face}$$

difference of 2 squares

7. Express  $\frac{(1+2i)^2}{1-i}$  in the form  $a+bi$ .

$$a^2 + 2ab + b^2$$

multiply above and  
below by the conjugate

Difference of 2 Squares

$$(1+2i)^2 = 1 + 4i + 4i^2 = -3 + 4i$$

$$\frac{(1+2i)^2}{1-i} = \frac{(-3+4i)(1+i)}{(1-i)(1+i)}$$

$$= \frac{-3 - 3i + 4i + 4i^2}{1 + 1i^2}$$

$$= \frac{-7 + i}{2}$$

$$= -\frac{7}{2} + \frac{1}{2}i$$

✓

8. Find the value of  $k$  if the real part of  $\frac{-3+i}{1+ki}$  is  $-3$ ,  $k \neq 0$ .

$$\begin{aligned} \frac{(-3+i)(1-ki)}{(1+ki)(1-ki)} &= \frac{-3+3ki+i+k^2i^2}{1+k^2} \\ &= \frac{(-3+k)+(3k+1)i}{1+k^2} = \frac{(-3+k)}{1+k^2} + \frac{(3k+1)i}{1+k^2} \\ \text{Re: Re} \Rightarrow \frac{-3+k}{1+k^2} &= -3 \\ \Rightarrow -3+k &= -3(1+k^2) \\ \Rightarrow -3+k &= -3-3k^2 \\ 3k^2+k &= 0 \\ k(3k+1) &= 0 \\ k \neq 0 & \quad | \quad 3k+1 = 0 \\ & \quad k = -\frac{1}{3} \quad \checkmark \end{aligned}$$

9. Simplify  $(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12})^2$ .

de Moivre

use calculator

$$\begin{aligned} \left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12}\right)^2 &= \left(\cos \frac{2\pi}{12} + i \sin \frac{2\pi}{12}\right) \\ &= \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\ \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) &= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \end{aligned}$$

$$\Rightarrow \text{original expression} = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}i\right) + \left(\frac{\sqrt{3}}{2}i\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2}i\right)$$

$$= \frac{\sqrt{3}}{4} + \frac{1}{4}i + \frac{3}{4}i + \frac{\sqrt{3}}{4}i^2$$

$$= \frac{1}{4}i \quad \checkmark$$

10. Show that  $1 + 2i$  is a root of the equation  $z^2 - (3 + 3i)z + 5i = 0$  and find the other root.

$$z^2 - (R_1 + R_2)z + (R_1 R_2) = 0$$

$$\text{Sum of Roots} = 3 + 3i$$

Let other root be  $x + yi$

Sum of Roots

$$\Rightarrow (1+2i) + (x+yi) = (3+3i)$$

Subtract  $(1+2i)$

$$\Rightarrow x + yi = 3 + 3i - 1 - 2i \\ = 2 + i \quad \checkmark$$

11. Simplify the following expression giving your answer in the form  $a + bi$ ,  $a, b \in \mathbb{R}$ :

$$(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})^2 (\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3})^4$$

de Moivre

$$(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})^2 = (\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3})$$

use calculator

$$= (-\frac{1}{2} + \frac{\sqrt{3}}{2}i)$$

de Moivre

$$(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3})^4 = (\cos 4(\frac{2\pi}{3}) + i \sin 4(\frac{2\pi}{3}))$$

use calculator

$$= (\cos 8\pi/3 + i \sin 8\pi/3) = (-\frac{1}{2} + \frac{\sqrt{3}}{2}i)$$

$$\Rightarrow \text{original expression} = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$= \left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}i\right) + \left(\frac{\sqrt{3}}{2}i\right)^2$$

$$= \frac{1}{4} - \frac{\sqrt{3}}{2}i + \frac{3}{4}i^2$$

$$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i \quad \checkmark$$

12. One of the roots of  $z^3 + z^2 + 4z + \rho = 0$ , where  $\rho$  is real, is  $1 - 3i$ . Find the value of  $\rho$  and the other two roots.

Conjugate root theorem

If  $1 - 3i$  is a root  $\Rightarrow 1 + 3i$  is a root ✓

quadratic factor =?

$$z^2 - (R_1 + R_2)z + (R_1 R_2) = 0$$

Difference of 2 squares

quadratic factor

$$\Rightarrow z^2 - 2z + 10 = 0$$

divide

Remainder = 0

$$\begin{array}{r} z+3 \\ \hline z^2 - 2z + 10 \end{array} \left. \begin{array}{r} z^3 + z^2 + 4z + \rho \\ - z^3 - 2z^2 - 10z \\ \hline 3z^2 - 6z + \rho \\ - 3z^2 - 6z - 70 \\ \hline \end{array} \right.$$

$$\Rightarrow \rho = 30 \quad \checkmark$$

$$\begin{array}{l} \text{FACTOR} \\ (z+3) = 0 \end{array} \Rightarrow \begin{array}{l} \text{ROOT} \\ z = -3 \quad \checkmark \end{array}$$

13. Given  $x + iy = \sqrt{8 - 6i}$ .

By squaring both sides of this equation, use simultaneous equations to find the values of  $x$  and  $y$ .

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$\operatorname{Re} = \operatorname{Re} \quad | \quad \operatorname{Im} = \operatorname{Im}$$

Solve from ②

$$\textcircled{2} \rightsquigarrow \textcircled{1}$$

multiply by  $y^2$ 

factorise

Since  $y \in \mathbb{R}$   
ignore imaginary solution

Solutions

$$(x+iy)^2 = 8 - 6i$$

$$x^2 + 2xyi + y^2 i^2 = 8 - 6i$$

$$x^2 - y^2 + 2xyi = 8 - 6i$$

$$x^2 - y^2 = 8 \quad \textcircled{1}$$

$$\Rightarrow x = \frac{-3}{y}$$

$$\Rightarrow (-\frac{3}{y})^2 - y^2 = 8$$

$$(\frac{9}{y^2}) - y^2 = 8$$

$$9 - y^4 = 8y^2$$

$$y^4 + 8y^2 - 9 = 0$$

$$(y^2 + 9)(y^2 - 1) = 0$$

$$y^2 = -9, \quad y^2 = 1$$

$$y = \pm 3, \quad y = \pm 1$$

$$\Rightarrow -3 + 1i = \sqrt{8 - 6i} \quad \text{and} \quad +3 - 1i = \sqrt{8 - 6i} \quad \checkmark$$

X values

$$x = -3/y$$

$$x = -3/x = -3$$

$$x = -3/1 = +3$$

*4<sup>th</sup> degree polynomial  
has 4 solutions*

$$\begin{aligned} i &= i \\ i^2 &= -1 \quad \text{If } t_i \text{ is solution} \\ i^3 &= -i \quad \Rightarrow f(t_i) = 0 \\ i^4 &= 1 \end{aligned}$$

14. Find the value of  $t$  for which  $ti$  is a solution of the equation  

$$z^4 - 2z^3 + 7z^2 - 4z + 10 = 0.$$
  
Hence find all the solutions of this equation.

$$\Rightarrow (t_i)^4 - 2(t_i)^3 + 7(t_i)^2 - 4(t_i) + 10 = 0$$

$$t^4(1) - 2(t^3)(-i) + 7(t^2)(-1) - 4ti + 10 = 0$$

$$t^4 + 2t^3i - 7t^2 - 4ti + 10 = 0 \quad \text{①}$$

$$Re = Re$$

$$\begin{aligned} t^4 - 7t^2 + 10 &= 0 \\ (t^2 - 5)(t^2 - 2) &= 0 \\ t^2 = 5, \quad t^2 = 2 & \\ t = \pm\sqrt{5}, \quad t = \pm\sqrt{2} & \end{aligned}$$

$$Im = Im$$

$t = \pm\sqrt{2}$  since it satisfies real and imaginary equations

$$\begin{aligned} 2t^3 - 4t &= 0 \\ t^3 - 2t &= 0 \\ t(t^2 - 2) &= 0 \\ t = 0, \quad t^2 = 2 &\Rightarrow t = \pm\sqrt{2} \end{aligned}$$

$$ti = \text{solution} \Rightarrow ti = \pm\sqrt{2}i \quad \checkmark$$

*4<sup>th</sup> degree polynomial  
has 4 solutions*

quadratic factor?

$$z^2 - (R_1 + R_2)z + (R_1 R_2) = 0$$

Divide

14. Find the value of  $t$  for which  $ti$  is a solution of the equation  

$$z^4 - 2z^3 + 7z^2 - 4z + 10 = 0.$$
  
Hence find all the solutions of this equation.

$$ti = \pm\sqrt{2}i$$

$$\begin{aligned} z^2 - (\sqrt{2}i - \sqrt{2}i)z + (\sqrt{2}i)(\sqrt{2}i) &= 0 \\ z^2 + 0z \pm 2i^2 &\Rightarrow z^2 + 0z + 2 = 0 \end{aligned}$$

$$\begin{array}{r} \frac{z^2 - 2z + 5}{z^2 + 0z + 2} \\ \hline z^4 - 2z^3 + 7z^2 - 4z + 10 \\ \underline{-z^4 - 0z^3 - 2z^2} \\ -2z^3 + 5z^2 - 4z \\ \underline{-2z^3 - 0z^2 - 4z} \\ 5z^2 + 0z + 10 \\ \underline{+5z^2 + 0z + 10} \end{array}$$

Solve quadratic

$$z^2 - 2z + 5 = 0 \quad a = 1, b = -2, c = 5$$

$$z = \frac{-b \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

4 Solutions

$$z = -\sqrt{2}i, \sqrt{2}i, 1 + 2i, 1 - 2i \quad \checkmark$$

15. Given that  $\bar{z}$  is the conjugate of  $z$  and  $z = a + bi$  where  $a$  and  $b$  are real, find the possible values of  $z$  if  $z\bar{z} - 2iz = 7 - 4i$ .

Difference 2 Squares

$$z = a + bi \Rightarrow \bar{z} = a - bi$$

$$z\bar{z} = a^2 + b^2 \quad \cancel{+ b^2 i^2} = a^2 + b^2$$

$$-2iz = -2i(a + bi) = -2ai \cancel{+ 2b i^2} = 2b - 2ai$$

$$\Rightarrow z\bar{z} - 2iz = a^2 + b^2 + 2b - 2ai = 7 - 4i$$

$$Im = Im$$

$$\Rightarrow -2a = -4 \Rightarrow a = 2$$

$$Re = Re$$

$$\Rightarrow a^2 + b^2 + 2b = 7$$

$$(2)^2 + b^2 + 2b = 7$$

$$b^2 + 2b - 3 = 0$$

$$(b - 1)(b + 3)$$

$$b = 1, b = -3$$

$$\Rightarrow a + bi = 2 + i \text{ or } 2 - 3i \quad \checkmark$$

*Note answer in book incorrectly has  $a = -2$*

16. Determine the real  $p$  and  $q$  for which  $(p + iq)^2 = 15 - 8i$ .

Hence solve the equation  $(1 + i)z^2 + (-2 + 3i)z - 3 + 2i = 0$

$$(p + iq)^2 = p^2 + 2pq i \cancel{+ q^2 i^2} = 15 - 8i$$

$$\Rightarrow p^2 - q^2 + 2pq i = 15 - 8i$$

$$Re = Re \quad | \quad Im = Im$$

$$p^2 - q^2 = 15 \quad \textcircled{1}$$

$$2pq = -8$$

$$pq = -4$$

$$q = -4/p \quad \textcircled{2}$$

$$\textcircled{2} \rightarrow \textcircled{1}$$

$$p^2 - \left(\frac{-4}{p}\right)^2 = 15$$

$$p^2 - \left(\frac{16}{p^2}\right) = 15$$

$$\times p^2$$

solve

But  $p$  is real  
 $p \neq \pm i$

$$q = -4/p$$

$$p^4 - 16 = 15p^2$$

$$q = -4/4$$

$$p^4 - 15p^2 - 16 = 0$$

$$q = -4/4$$

$$(p^2 - 16)(p^2 + 1) = 0$$

$$q = -1$$

$$p^2 = 16$$

$$q = -4/-4$$

$$p = \pm 4$$

$$q = 1$$

Solution

$$\Rightarrow p + qi = 4 - i \quad \text{or} \quad -4 + i$$

$\checkmark$

16. Determine the real  $p$  and  $q$  for which  $(p + iq)^2 = 15 - 8i$ .  
 Hence solve the equation  $(1 + i)z^2 + (-2 + 3i)z - 3 + 2i = 0$

expand

$Re = Re$	$Im = Im$	$z^2 + z^2 i - 2z + 3zi - 3 + 2i = 0$ $z^2 - 2z - 3 = 0$ $(z - 3)(z + 1) = 0$ $z = 3, z = -1$	$z^2 + 3z + 2 = 0$ $(z + 1)(z + 2) = 0$ $z = -1, z = -2$
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$\Rightarrow z = -1$  is solution to original equation  
 Since it satisfies the real and imaginary equations  
 $\Rightarrow (z+1)$  is a factor

Divide

$$\begin{array}{r} (1+i)z + (-3+2i) \\ \hline z+1 \Big) (1+i)z^2 + (-2+3i)z - 3+2i \\ \quad \cancel{(1+i)z^2} \cancel{+ (1+i)z} \\ \quad \cancel{(-3+2i)z} - \cancel{3+2i} \\ \quad \cancel{(-3+2i)z} \cancel{+ 3+2i} \end{array}$$

Factor  
↓  
solution

$$(1+i)z + (-3+2i) = 0$$

$$z = \frac{-(3+2i)}{(1+i)} = \frac{(+3-2i)(1-i)}{(1+i)(1-i)} = \frac{3-3i-2i+2i^2}{1+1i^2} = \frac{1-5i}{2}$$

2 Solutions

$$z = -1, z = \frac{1}{2} - \frac{5}{2}i \quad \checkmark$$