

# Co-ordinate Geometry

One, possibly two, of the short questions (25 marks each) in Section A are likely to cover co-ordinate geometry of the line and the circle. It is also possible that co-ordinate geometry could occur in one of the long questions in Section B, although there is little evidence of this from the exam papers to date.

In Section A, there are likely to be question parts which test both your understanding of key concepts such as the slope of a line, and your ability to use the formulae and the methods on the course. In Section B, you may very well have to be inventive in how you set up your co-ordinate system and in what approach you use.

With co-ordinate geometry, it is important to remember that tougher questions often involve using ideas from synthetic geometry. You should be prepared for this.

## A. The Line

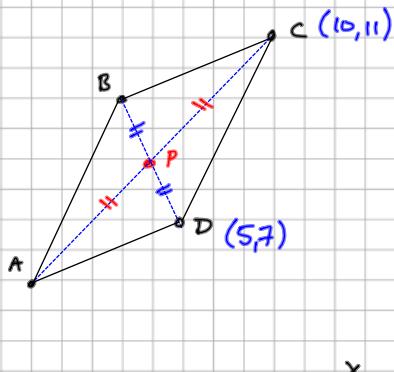
### 1. Basic concepts

e.g.  $A(2,3)$  and  $B(5,9)$  are two of the vertices of the parallelogram  $ABCD$ , and the diagonals of the parallelogram intersect at  $P(6,7)$ . Find the co-ordinates of  $C$  and  $D$ .

$P$  is midpt of  $[AC]$   
 Translation  
 $A(2,3) \xrightarrow{+4,+4} P(6,7) \xrightarrow{} C(10,11)$

$P$  is the midpt of  $[BD]$   
 Translation  
 $B(5,9) \xrightarrow{+1,-2} P(6,7) \xrightarrow{} D(7,5)$

Sketch



## Coordinate geometry

### 2. Area of a triangle

e.g. The area of the triangle  $OAB$ , where  $O = (0,0)$ ,  $A = (x, 6)$  and  $B = (6, 2)$ , is 14 square units. Find the possible values of  $x$ .

$$\Delta_{(0,0)} = \frac{1}{2} |x_1 y_2 - x_2 y_1|$$

2 options: if  $\oplus$

if  $\ominus$

$$\frac{1}{2} |x(2) - 6(6)| = 14$$

$$|2x - 36| = 28$$

$$2x - 36 = +28$$

$$2x = 64$$

$$x = 32$$

$$2x - 36 = -28$$

$$2x = 12$$

$$x = 6$$

\* could square both sides  
and solve quadratic

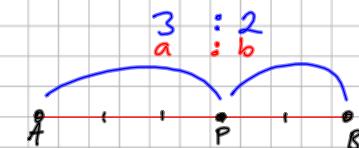
### 3. Divisor of a line segment

e.g. If  $A = (2, 8)$  and  $B = (-8, 13)$ , find the co-ordinates of the point which divides  $[AB]$  in the ratio  $3:2$ .

Point dividing a line segment in the ratio  $a:b$

$$P = \left( \frac{bx_1 + ax_2}{a+b}, \frac{by_1 + ay_2}{a+b} \right)$$

$A(2, 8)$        $B(-8, 13)$



$$\begin{aligned}
 P &= \left( \frac{2(2) + 3(-8)}{3+2}, \frac{2(8) + 3(13)}{3+2} \right) \\
 &= \left( \frac{4 - 24}{5}, \frac{16 + 39}{5} \right) \\
 &= \left( \frac{20}{5}, \frac{55}{5} \right) \\
 &= (4, 11)
 \end{aligned}$$

## Coordinate geometry

### 4. Angle between two lines

e.g. find, correct to the nearest degree, the larger angle between the lines  
 $2x - 3y = 7$  and  $3x + 5y + 9 = 0$

$$m = \frac{-a}{b}$$

$$L_1 : 2x - 3y - 7 = 0$$

$$m_1 = \frac{-2}{-3} = \frac{2}{3}$$

$$L_2 : 3x + 5y + 9 = 0$$

$$m_2 = -\frac{3}{5}$$

$$\tan \theta = \pm \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\tan \theta = \pm \sqrt{\frac{\left(\frac{2}{3}\right) - \left(-\frac{3}{5}\right)}{1 + \left(\frac{2}{3}\right)\left(-\frac{3}{5}\right)}}$$

$$\tan \theta = \pm \frac{19}{9}$$

acute angle  $\oplus$

$$\theta_1 = \tan^{-1}\left(\frac{19}{9}\right) \approx 64.6^\circ$$

larger angle

$$\theta_2 = 180 - 64.6 = 115.4^\circ$$

### 5. Perpendicular distance formula

e.g. if the perpendicular distance between the parallel lines  $x + 2y = 8$  and  $2x + 4y + k = 0$  is  $\sqrt{5}$ , find the two possible values of  $k \in \mathbb{R}$ .

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Pick any pt. on  $L_1$

$$L_1 : x + 2y = 8$$

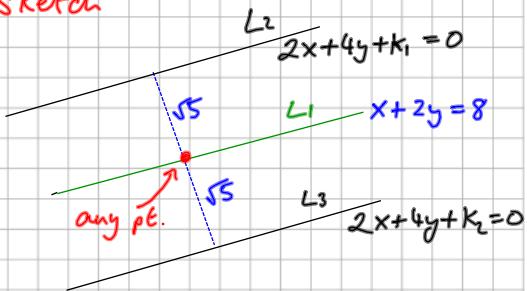
$$x = 0 \Rightarrow 2y = 8$$

$$\Rightarrow y = 4$$

$$pt (0, 4) \in L_1$$

$$x_1, y_1$$

Sketch



$$\Rightarrow \sqrt{5} = \frac{|2(0) + 4(4) + k|}{\sqrt{2^2 + 4^2}}$$

$$\sqrt{5} = \frac{|16 + k|}{\sqrt{20}}$$

$$\Rightarrow 10 = |16 + k|$$

$$10 = 16 + k \Rightarrow k_1 = -6$$

$$-10 = 16 + k \Rightarrow k_2 = -26$$

2 options:

$\oplus$

$\ominus$

## B. The Circle

1. Equation:  $x^2 + y^2 = r^2$

e.g. find the equation of the circle, with centre  $(0,0)$  and which has the line  $x - 2y = 15$  as a tangent

$$\begin{aligned} x - 2y &= 15 \\ x = 0 &\Rightarrow y = -7.5 \\ y = 0 &\Rightarrow x = 15 \end{aligned}$$

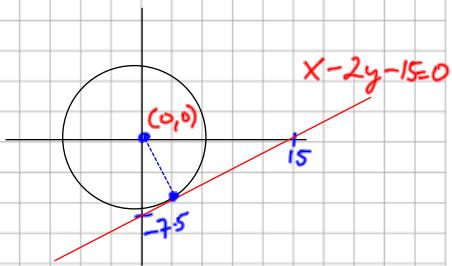
$\perp$  distance from line to centre  $(0,0)$   
= Radius

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

equation:

$$x^2 + y^2 = R^2$$

Sketch



$$d = \frac{|1(0) - 2(0) - 15|}{\sqrt{1^2 + 2^2}}$$

$$d = \frac{15}{\sqrt{5}} = R$$

$$x^2 + y^2 = \left(\frac{15}{\sqrt{5}}\right)^2$$

$$x^2 + y^2 = 45$$

2. Equation:  $(x-h)^2 + (y-k)^2 = r^2$

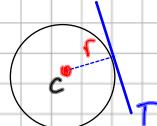
e.g. find the equation of the circle that has centre  $(-1,3)$  and has the line  $4x + y = 16$  as a tangent

$\perp$  distance from line  
 $4x + y - 16 = 0$  to  
centre  $(-1,3)$  is Radius

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

equation

$$(x-h)^2 + (y-k)^2 = R^2$$



$$R = \frac{|4(-1) + 1(3) - 16|}{\sqrt{4^2 + 1^2}}$$

$$R = \frac{17}{5}$$

$$(x - -1)^2 + (y - 3)^2 = \left(\frac{17}{5}\right)^2$$

$$(x + 1)^2 + (y - 3)^2 = \frac{289}{25}$$

## Coordinate geometry

3. Equation:  $x^2 + y^2 + 2gx + 2fy + c = 0$

e.g. the length of the radius of the circle

$$x^2 + y^2 - 2x + 2ky - 15 = 0$$

is 5; find the value of  $k > 0$  and the co-ordinates of the centre

$$\begin{aligned} 2g &= -2 \Rightarrow g = -1 \\ 2f &= 2k \Rightarrow f = k \\ c &= -15 \end{aligned}$$

$k = ?$

$$R = \sqrt{g^2 + f^2 - c}$$

$k > 0$

$$R = \sqrt{1^2 + k^2 + 15} = 5$$

$$\begin{aligned} 1 + k^2 + 15 &= 25 \\ k^2 &= 9 \\ k &= 3 \end{aligned}$$

Centre?

$$C = (-g, -f)$$

$$C = (1, -k)$$

$$C = (1, -3)$$

4.  $g, f, c$  method

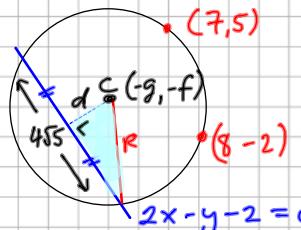
e.g. find the equation of the circle which contains the points  $(7, 5)$  and  $(8, -2)$  if the line  $2x - y - 2 = 0$  contains a chord of length  $4\sqrt{5}$  of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

we want 3 equations!

sub  $\left\{ \begin{array}{l} \text{contains } (7, 5) \\ \text{contains } (8, -2) \end{array} \right.$

Sketch



$$(7)^2 + (5)^2 + 2g(7) + 2f(5) + c = 0$$

$$49 + 25 + 14g + 10f + c = 0$$

$$14g + 10f + c = -74 \quad (1)$$

$$(8)^2 + (-2)^2 + 2g(8) + 2f(-2) + c = 0$$

$$64 + 4 + 16g - 4f + c = 0$$

$$16g - 4f + c = -68 \quad (2)$$

Since 1 line from centre to chord bisects chord

$C(-g, -f)$

$d$

$2\sqrt{5}$

$2x - y - 2 = 0$

$R = \sqrt{g^2 + f^2 - c}$

also  $d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

$d = \sqrt{R^2 - 20}$

$d = \sqrt{g^2 + f^2 - c - 20}$

$d = \sqrt{R^2 - 20}$

$$\Rightarrow \sqrt{(\sqrt{g^2 + f^2 - c})^2 - 20} = \frac{|2(-g) - 1(-f) - 2|}{\sqrt{2^2 + 1^2}}$$

$$g^2 + f^2 - c - 20 = \frac{(-2g + f - 2)^2}{5}$$

$$5g^2 + 5f^2 - 5c - 100 = 4g^2 - 2gf + 4g$$

$$-2gf + f^2 - 2f + 4g - 2f + 4$$

$$g^2 + 4f^2 + 4gf + 4f - 8g - 5c - 104 = 0 \quad (3)$$

## Coordinate geometry

<p>Solve 3 equations</p> $\textcircled{1} \rightarrow c = -16g + 4f - 68$ $\textcircled{2} - \textcircled{1}$ $\textcircled{4} \rightarrow \textcircled{1}$ $c = -16(7f+3) + 4f - 68$ $= -112f - 48 + 4f - 68$ $c = -108f - 116 \quad \textcircled{5}$ $\textcircled{4} + \textcircled{5} \rightarrow \textcircled{3}$ <p>Centre = <math>(-g, -f)</math> Centre = <math>(4, 1)</math></p> <p>Radius = <math>\sqrt{g^2 + f^2 - c}</math> Radius = <math>\sqrt{4^2 + 1^2 - 8} = 3</math></p>	$14g + 10f + c = -74 \quad \textcircled{1}$ $16g - 4f + c = -68 \quad \textcircled{2}$ $g^2 + 4f^2 + 4gf + 4f - 4g - 5c - 104 = 0 \quad \textcircled{3}$ $2g - 14f = 6$ $g - 7f = 3, g = 7f + 3 \quad \textcircled{4}$ $(7f+3)^2 + 4f^2 + 4(7f+3)f + 4f - 8(7f+3) - 5(108f - 116) - 104 = 0$ <p>Solve quadratic</p> $49f^2 + 42f + 9 + 4f^2 + 28f^2 + 12f - 4f - 56f - 24 + 540f + 580 - 104 = 0$ $81f^2 + 542f + 461 = 0$ $(81f + 461)(f + 1) = 0$ $f = -\frac{461}{81} \text{ or } f = -1$ $g = 7(-1) + 3 = -7 + 3 = -4$ $c = -108(-1) - 116 = -8$ $c = -8$ <p>Circle equation:  <math>x^2 + y^2 + 2gx + 2fy + c = 0</math></p> $x^2 + y^2 - 8x - 2y - 8 = 0$
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5. Touching circles  
e.g. investigate if the circles

$S_1: x^2 + y^2 - 8x + 2y + 12 = 0$  and  
 $S_2: x^2 + y^2 - 14x - 10y + 54 = 0$

intersect at a single point

Is distance between centres =  $R_1 + R_2$ ?

$C = (-g, -f)$   
 $C_1 = (4, -1)$   
 $C_2 = (7, 5)$

$d = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}$

$R = \sqrt{g^2 + f^2 - c}$

$R_1 = \sqrt{4^2 + 1^2 - 12} = \sqrt{5}$   
 $R_2 = \sqrt{7^2 + 5^2 - 54} = 2\sqrt{5}$

distance between centres

$$d = \sqrt{(7-4)^2 + (5-(-1))^2}$$

$$d = \sqrt{45} = 3\sqrt{5}$$

Sum of radii

$$R_1 + R_2 = \sqrt{5} + 2\sqrt{5} = 3\sqrt{5}$$

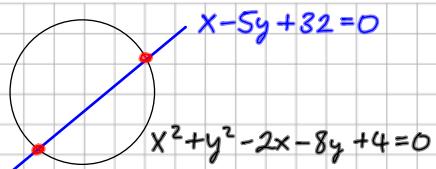
They intersect at a single point.

## Coordinate geometry

### 6. Intersection of a line and a circle

e.g. find the points of intersection of the line  $x - 5y + 32 = 0$  and the circle

$$x^2 + y^2 - 2x - 8y + 4 = 0$$



① Rewrite linear

$$X = 5y - 32$$

② Sub into circle  
and solve

$$(5y - 32)^2 + y^2 - 2(5y - 32) - 8y + 4 = 0$$

$$25y^2 - 320y + 1024 + y^2 - 10y + 64 - 8y + 4 = 0$$

$$26y^2 - 338y + 1092 = 0$$

$$y^2 - 13y + 42 = 0$$

$$(y - 6)(y - 7) = 0$$

$$y = 6 \quad \text{or} \quad y = 7$$

③ Sub back into linear

$$y = 6 \quad X = 5(6) - 32 = -2$$

$$y = 7 \quad X = 5(7) - 32 = 3$$

Points of intersection:  $(-2, 6)$  and  $(3, 7)$

### 7. Tangent at a point

e.g. find the equation of the tangent to the circle  $x^2 + y^2 + 4x - 2y - 20 = 0$  at the point  $(2, 4)$

$$C = (-g, -f)$$

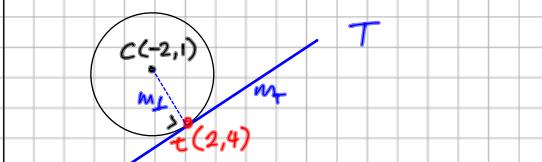
$$C = (-2, 1)$$

$$\text{slope, } m = \frac{y_2 - y_1}{x_2 - x_1}$$

equation:

$$y - y_1 = m(x - x_1)$$

Tangent:



slope  $[c:t]$

$$m_1 = \frac{4 - 1}{2 - (-2)} = \frac{3}{4}$$

perpendicular slope = slope of T

$$m_T = -\frac{4}{3}$$

equation of T with  $(2, 4)$

$$y - 4 = -\frac{4}{3}(x - 2)$$

$$3y - 12 = -4x + 8$$

$$4x + 3y - 20 = 0$$

## Coordinate geometry

### 8. Tangents and chords

e.g. find the equations of the tangents that can be drawn from the point  $(0, -9)$  to the circle

$$x^2 + y^2 + 2x - 8y + 7 = 0.$$

$$C = (-g, -f)$$

$$C = (-1, 4)$$

$$R = \sqrt{g^2 + f^2 - c}$$

$$R = \sqrt{1^2 + 4^2 - 7} = \sqrt{10}$$

equation:

$$y - y_1 = m(x - x_1)$$

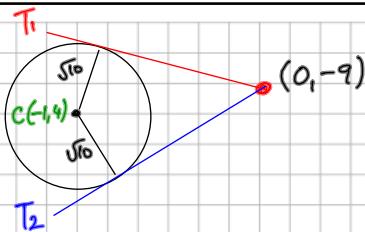
distance from pt. to line

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$\text{Tangents } \begin{cases} m = 53/9 \\ m = -3 \end{cases} \Rightarrow$$

$$y = \frac{53}{9}x - 9$$

$$y = -3x - 9$$



Tangent with  $(0, -9)$   
or  $y + 9 = m(x - 0) \Rightarrow y = mx - 9$

Distance from centre to tangent = radius

$$\sqrt{10} = \frac{|m(-1) - 1(4) - 9|}{\sqrt{m^2 + 1^2}}$$

$$(\sqrt{10}\sqrt{m^2 + 1})^2 = (-m - 13)^2$$

$$10(m^2 + 1) = m^2 + 26m + 169$$

$$9m^2 - 26m - 159 = 0$$

$$(9m - 53)(m + 3) = 0$$

$$m = \frac{53}{9} \quad \text{OR} \quad m = -3$$