Friday 14 March (9:00 - 11:00am)

Part 1: Introduction to Differentiation

1. Product Rule
2. Quotient Rule
3. Chain Rule
4. Slopes & Tangents
5. Turning Points

OL 2007
Q7 6(i)

Product rule

$$y = uv \quad \Rightarrow \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$u = x^2 + 9$

$\frac{du}{dx} = 2x$

$v = 4x^3 + 5$

$\frac{dv}{dx} = 12x^2$

$f(x) = (x^2+9)(4x^3+5)$

$$f'(x) = (x^2+9)(12x^2) + (4x^3+5)(2x)$$
$$= 12x^4 + 108x^2 + 8x^4 + 10x$$
$$= 20x^4 + 108x^2 + 10x$$

Quotient rule

$$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$u = 3x$

$\frac{du}{dx} = 3$

$v = 2x+3$, $v^2 = (2x+3)^2$

$\frac{dv}{dx} = 2$

OL 2007 7 b ii

$y = \frac{3x}{2x+3}$

$$\frac{dy}{dx} = \frac{(2x+3)(3) - (3x)(2)}{(2x+3)^2}$$

$$= \frac{6x+9-6x}{(2x+3)^2}$$

$$= \frac{9}{(2x+3)^2}$$

Chain Rule

"fn of fn of x"

$\frac{dy}{dx} = ?$

"Derivative of outside × Derivative of inside"

$\frac{dy}{dx}_{(x=2)} = ?$

OL 2006 Q7 b ii

$y = ((5-x^2))^3$

$\frac{dy}{dx} = 3((5-x^2))^2 . (-2x)$

$\frac{dy}{dx} = -6x(5-x^2)^2$

$\frac{dy}{dx}_{(x=2)} = -6(2)(5-2^2)^2$

$= -12(1)^2$

$= -12$

Chain Rule

OL 2004
Q7 b(ii)

$y = (x^2 - 2x - 3)^3$

Show that $\frac{dy}{dx} = 0$ when $x = 1$

" Derivative of outside
× Derivative of inside "

$\frac{dy}{dx} = 3(x^2 - 2x - 3)^2 \cdot (2x - 2)$

$= (6x - 6)(x^2 - 2x - 3)^2$

$x = 1$ $\Rightarrow \frac{dy}{dx} = (6(1) - 6)(1^2 - 2(1) - 3)^2$

$= 0$ QED

Chain Rule

OL 1998
Q7 b(ii)

$y = (4 - 3x^2)^7$

$\frac{dy}{dx} = ?$ and range of values of x, $\frac{dy}{dx} > 0$

" Derivative of outside
× Derivative of inside "

$\frac{dy}{dx} = 7(4 - 3x^2)^6(-6x)$

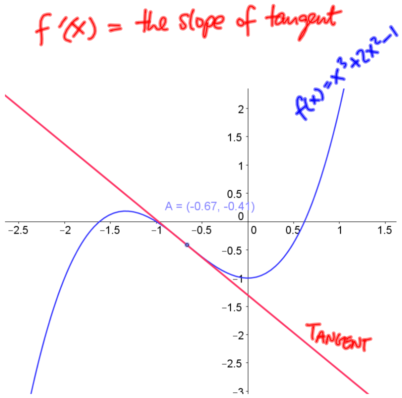
$\frac{dy}{dx} = -42x(4 - 3x^2)^6$

$\frac{dy}{dx} > 0$ $\Rightarrow \underbrace{-42x}_{x<0}\underbrace{(4 - \overset{+}{3x^2})^6}_{\geq 0} > 0$

$x < 0$

Tangents

$f'(x) = $ the slope of tangent



$f(x) = x^3 + 2x^2 - 1$

(i) $f'(x) = ?$

$f'(x) = 3x^2 + 4x$

(ii) L is Tangent @ $x = -\frac{2}{3}$

$\boxed{y - y_1 = m(x - x_1)}$

slope

$m = f'(-\frac{2}{3}) = 3(-\frac{2}{3})^2 + 4(-\frac{2}{3})$

$m = -4/3$

point? $x = -\frac{2}{3}$, $y = ?$

$f(-\frac{2}{3}) = (-\frac{2}{3})^3 + 2(-\frac{2}{3})^2 - 1 = -\frac{11}{27}$

pt $(-2/3, -11/27)$

$L: y - -11/27 = -4/3\,(x - -\frac{2}{3})$

$3(y + 11/27) = -4(x + 2/3)$

$3y + 33/27 = -4x - 8/3$

$+4x + 3y + 35/9 = 0$

$L: 36x + 27y + 35 = 0$

Find Tangent of $f(x)$ when $x=a$

- Differentiate : $f'(x) = ?$
- Find slope : $f'(a) = ?$
- Get point : $f(a) = ?$ $(a, f(a))$
- Get tangent equation : $y - y_1 = m(x - x_1)$

Tangent?

OL 2007
Q6 c

$f'(x) = ?$

$$f(x) = (5x-2)^4$$

$$f'(x) = 4(5x-2)^3.(5)$$

$$f'(x) = 20(5x-2)^3$$

Slope $= 20$
i.e.. $f'(x) = 20$

$\div 20$

Cube Root

$$20(5x-2)^3 = 20$$

$$(5x-2)^3 = 1$$

$$5x-2 = 1$$

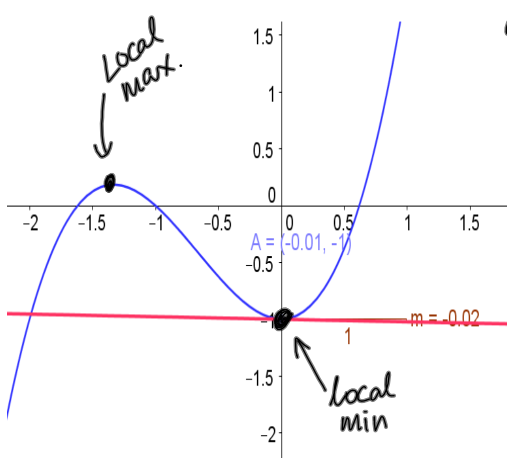
$$5x = 3 \quad \Rightarrow x = 3/5$$

$x = 3/5$
$y = ?$

$$f\left(\tfrac{3}{5}\right) = \left(5\left(\tfrac{3}{5}\right)-2\right)^4 = (3-2)^4 = 1^4 = 1$$

pt $(3/5, 1)$

Find Turning Points



at local max
and local min

$$\boxed{f'(x) = 0}$$

To find turning points

- Differentiate $f'(x) = ?$
- Slope $= 0 \Rightarrow \boxed{f'(x) = 0}$ Solve for xs
- Sub into function to get ys.

Turning point

OL 2005 Q6 c

$f(x) = x^2 + px + 10$

(i) $f'(x) = ?$

$f'(x) = 2x + p$

(ii) $x = 3$ at min, find p?

at min $\Rightarrow f'(x) = 0$ $\Rightarrow 2x + p = 0$

$x = 3$ $\Rightarrow 2(3) + p = 0 \Rightarrow p = -6$

we now have $f(x)$ and $f'(x)$

$f(x) = x^2 - 6x + 10$

$f'(x) = 2x - 6$

pt $(0,10)$ tangent?

Slope $= f'(0) = 2(0) - 6 = -6$

$y - y_1 = m(x - x_1)$

$y - 10 = -6(x - 0)$

$6x + y - 10 = 0$

Turning points

OL 1998 Q6 c

$f(x) = (x+k)(x-2)^2$, $k \in \mathbb{R}$

(i) $f(3) = 7$, $k = ?$

$\Rightarrow (3+k)(3-2)^2 = 7$

$(3+k)(1)^2 = 7$

$3 + k = 7$

$k = 4$

(ii) Local max and min ?

$f(x) = \boxed{(x+4)}\boxed{(x-2)^2}$

$= (x+4)(x^2 - 4x + 4)$

$= x^3 - \cancel{4x^2} + 4x + \cancel{4x^2} - 16x + 16$

$= x^3 - 12x + 16$

at max/min $\Rightarrow f'(x) = 0$

differentiate: $f'(x) = 3x^2 - 12$

at max/min $\Rightarrow 3x^2 - 12 = 0$

$x^2 - 4 = 0$

$x^2 = 4$

$x = \pm 2$

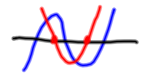
$x = 2$: $f(2) = (2)^3 - 12(2) + 16 = 8 - 24 + 16 = 0$

pt $(2, 0)$

$x = -2$: $f(-2) = (-2)^3 - 12(-2) + 16 = -8 + 24 + 16 = 32$

pt $(-2, 32)$

Differentiation Part 2: 11:40 - 1:00 pm

1. Rates of change $\frac{d}{dt}$
2. Graphs of derivatives 
3. Higher order differentiation

   1st: $\frac{dy}{dx}$, $f'(x)$

   2nd: $\frac{d^2y}{dx^2}$, $f''(x)$
4. Points of inflection $f''(x)=0$

"Slope"

$$\frac{dy}{dx} = \text{"Rate of change"}$$

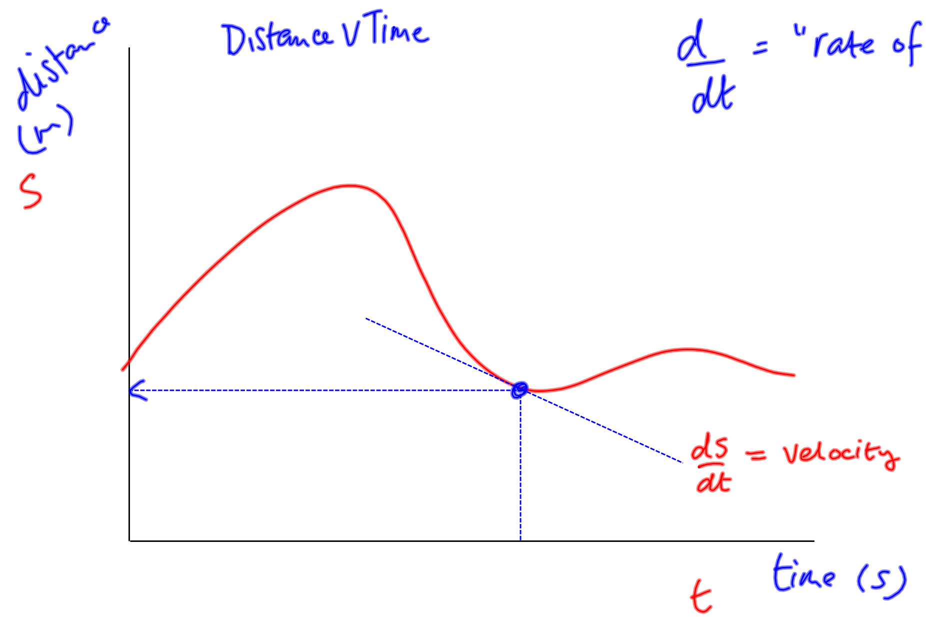
"division"

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{(x+h)-(x)}$$

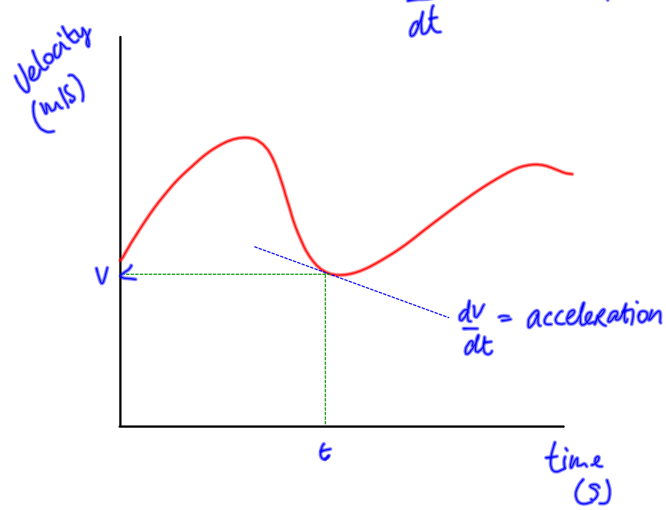
output

input

$\frac{d}{dt}$ = "rate of change"



$\frac{d}{dt}$ = "rate of change"

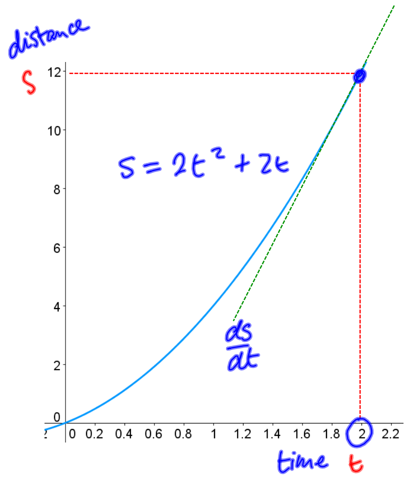


In Summary

S = distance
V = velocity
a = acceleration

$$\boxed{\begin{aligned} V &= \frac{ds}{dt} \\ a &= \frac{dv}{dt} = \frac{d^2 s}{dt^2} \end{aligned}}$$

2nd derivative



2007 Q7 c

(distance) $S = 2t^2 + 2t$

(Velocity) $V = \frac{ds}{dt} = 4t + 2$

$V = 4t + 2$

(acceleration) $a = \frac{d^2s}{dt^2} = \frac{dv}{dt} = 4$

$a = 4$

(i) $t = 2$, $V = ?$ $V = 4t + 2 = 4(2) + 2 = 10$ m/s

(ii) $a = 4$ m/s²

(iii) $S = 24$ m $t = ?$ $\Rightarrow 2t^2 + 2t = 24$

$t^2 + t - 12 = 0$

$(t - 3)(t + 4) = 0$

$t = 3$, $\cancel{t = -4}$

$t = 3$ s

2001 Q7 (c)

$h = 80t - 5t^2$

(height) or distance: $h = 80t - 5t^2$

Velocity: $\frac{dh}{dt} = V = 80 - 10t$

acceleration: $\frac{dv}{dt} = a = -10$

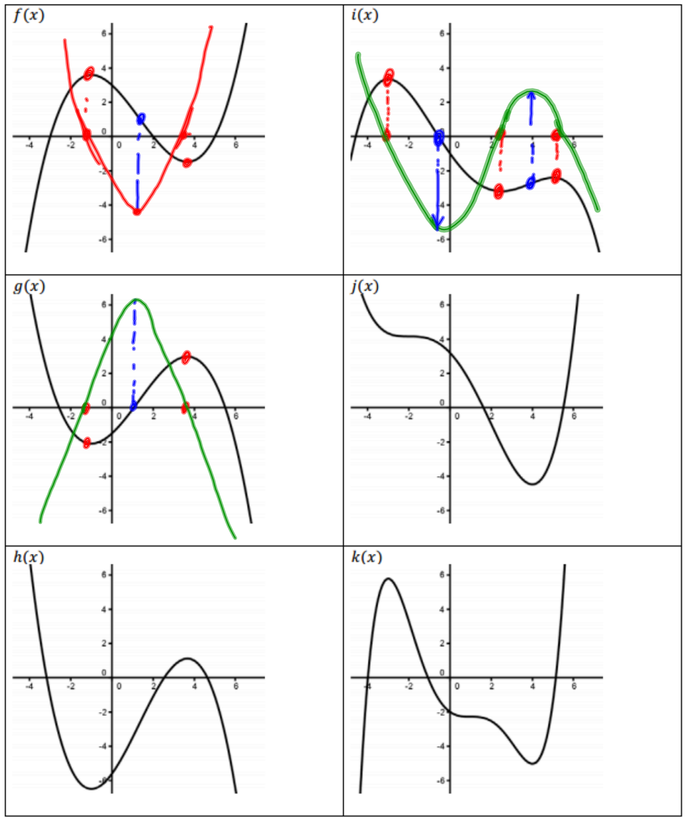
(i) $t = 5$ $h = ?$ $h = 80(5) - 5(5)^2 = 275$ m

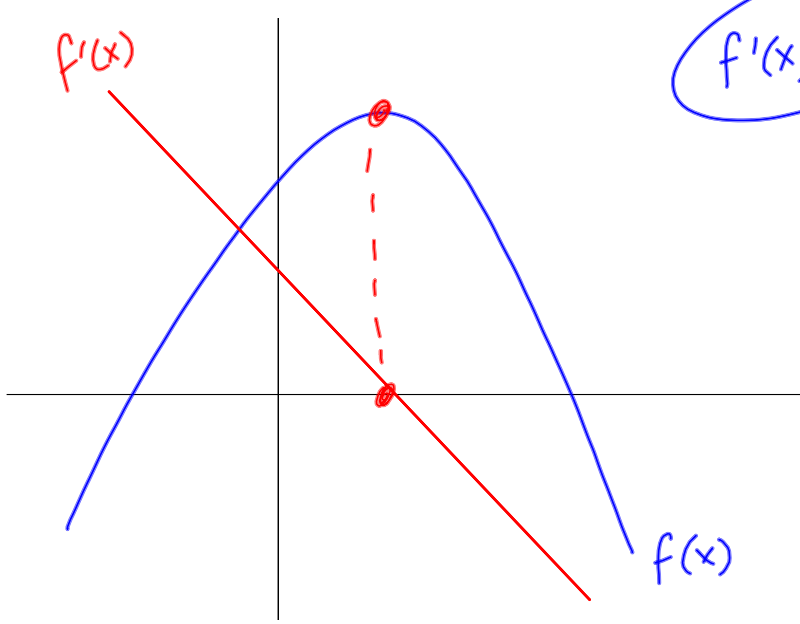
(ii) $t = 5$ $V = ?$ $V = 80 - 10(5) = 30$ m/s

at max $\frac{dh}{dt} = V = 0$

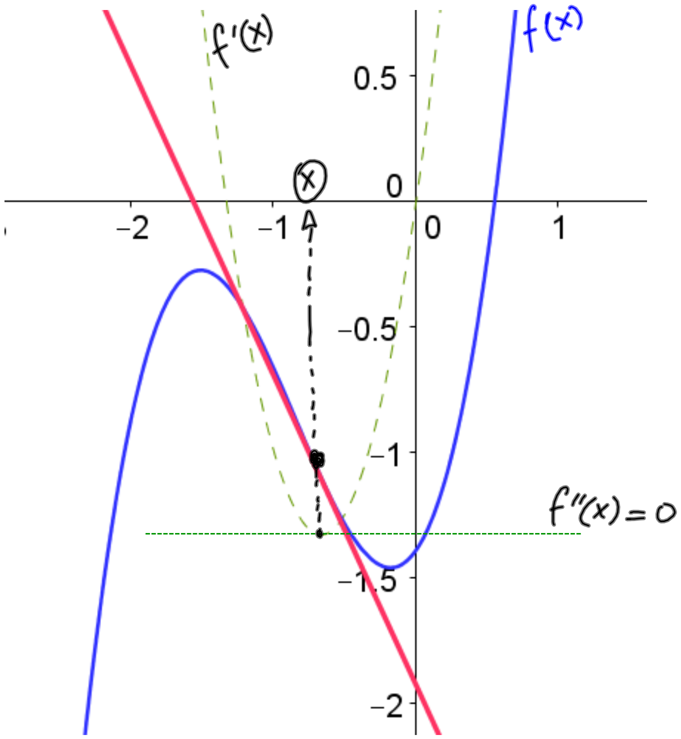
(iii) $h = \max$, $t = ?$ $V = 80 - 10t = 0$

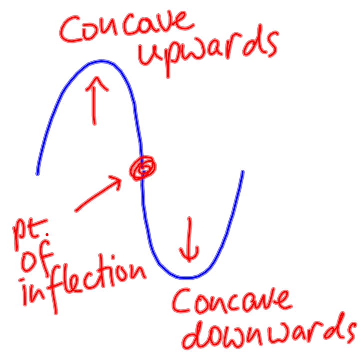
$80 = 10t \Rightarrow t = 8$ s





$f'(x)$?

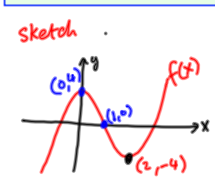




2002

6 (c) The function $f(x) = ax^3 + bx^2 + cx + d$ has a maximum point at (0, 4) and a point of inflection at (1, 0). Find the values of a, b, c and d.

sketch



derivatives:

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

$(0,4) \to (1,0) \to (2,-4)$

at max, $f'(x) = 0$

$\Rightarrow 3ax^2 + 2bx + c = 0$

at max $x = 0$ $\quad \Rightarrow 0 + 0 + c = 0 \Rightarrow c = 0$ ①

at inflection pt. $f''(x) = 0$

inflection $x = 1$ $\quad \Rightarrow 6ax + 2b = 0 \Rightarrow 6a + 2b = 0 \Rightarrow 3a + b = 0$ ②

pt $(0,4) \Rightarrow f(0) = 4$ $\quad \Rightarrow d = 4$ ③

pt $(1,0) \Rightarrow f(1) = 0$ $\quad \Rightarrow a(1)^3 + b(1)^2 + 0 + 4 = 0$

$a + b = -4$ ④

② − ④ $\Rightarrow 2a = 4 \Rightarrow a = 2$

→ ④ $\quad 2 + b = -4 \Rightarrow b = -6$

$f(x) = 2x^3 - 6x^2 + 4$