# Functions Revision

A function is a relationship that maps each input onto a single output.

**1. Definition of a function**

e.g. The function $g : X \to Y : x \to \dfrac{2}{x+1}$ is defined on the set $X = \{1, 2, 3, 4, 5\}$.

DOMAIN

List the elements that $Y$ must contain.

$g(1) = \dfrac{2}{1+1} = \dfrac{2}{2} = 1$

$g(2) = \dfrac{2}{2+1} = \dfrac{2}{3}$

$g(3) = \dfrac{2}{3+1} = \dfrac{2}{4} = \dfrac{1}{2}$

$g(4) = \dfrac{2}{4+1} = \dfrac{2}{5}$

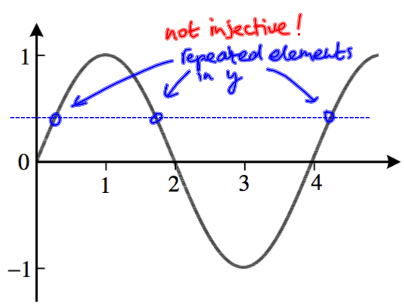
$g(5) = \dfrac{2}{5+1} = \dfrac{2}{6} = \dfrac{1}{3}$

y-Values (Range)

$1, \ 2/3, \ 1/2, \ 2/5, \ 1/3$

**2. Types of functions**

e.g. The curve shown below represents part of a function $f : \mathbb{R} \to \mathbb{R}$. The shape of the curve continues as shown in both direction.




(i) Use the diagram to explain if $f$ is injective.

(ii) State the range of the function, and use it to explain if $f$ is surjective.

(iii) If we restrict the domain to $[1,3]$ and the codomain to $[-1,1]$, explain why the curve now represents a bijective function.

Injective → no repeated outputs

Surjective → all y-values are mapped 'onto'

Bijective → Injective & Surjective

(i) Its not Injective – It fails the horizontal line test – not all output values are unique.

(ii) It seems to be a 'sinusidal' wave with a Range $[-1, 1]$ ⇒ Its not surjective – as not all elements in y have been mapped onto eg. no input gives $y = 2$ or $y = -3$, or any other value outside the small Range

(iii) If domain restricted to $[1,3]$ then it is injective
If Codomain is restricted to $[-1,1]$ then it is surjective
⇒ its bijective

**3. Composite functions**

e.g. $f : x \to (x-1)^2$ and $g : x \to 3x+1$ are two functions defined for all $x \in \mathbb{R}$.

(i) Investigate if $g \circ f(x) = f \circ g(x)$, for all $x \in \mathbb{R}$.

(ii) Express $g^2(x)$ in terms of $x$.

$g \circ f(x) = g(f(x))$ "g after f"

$$g \circ f(x) = 3[(x-1)^2] + 1 = 3(x^2 - 2x + 1) + 1 = 3x^2 - 6x + 3 + 1 = 3x^2 - 6x + 4$$

$f \circ g(x) = f(g(x))$ "f after g"

$$f \circ g(x) = ((3x+1)-1)^2 = (3x)^2 = 9x^2$$

$\Rightarrow \quad f \circ g \neq g \circ f$

$g^2 = g(g(x))$ "g after g"

$$g^2(x) = 3(3x+1)+1 = 9x + 3 + 1 = 9x + 4$$

**4. Inverse functions**

e.g. $f$ is the function $f : x \to \dfrac{5x-2}{2x+3}$.

(i) Find $f^{-1}(x)$.

(ii) Verify that $f^{-1} \circ f(x) = x$.

Steps:
1. write $y = \ldots$
2. rearrange so $x = \ldots$
3. exchange $x$ for $f^{-1}(x)$ and $y$ for $x$

(i) Inverse function ?

1. $y = \dfrac{5x-2}{2x+3}$

2. rearrange

$$y(2x+3) = 5x - 2$$
$$2yx + 3y = 5x - 2$$
$$5x - 2yx = 3y + 2$$
$$x(5 - 2y) = 3y + 2$$
$$x = \frac{3y+2}{5-2y}$$

exchange 3. $f^{-1}(x) = \dfrac{3x+2}{5-2x}$

(ii) $f^{-1} \circ f(x)$

$$= \frac{3\left(\frac{5x-2}{2x+3}\right) + 2}{5 - 2\left(\frac{5x-2}{2x+3}\right)} = \frac{\frac{3(5x-2)+2(2x+3)}{\cancel{2x+3}}}{\frac{5(2x+3)-2(5x-2)}{\cancel{2x+3}}}$$

$$= \frac{15x \cancel{-6} + 4x \cancel{+6}}{\cancel{10x} + 15 \cancel{-10x} + 4} = \frac{19x}{19} = x$$

QED