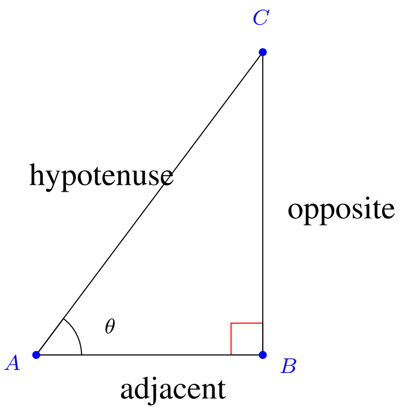
Trigonometry — What do you need to know?

1. The definitions of $\sin x$, $\cos x$ and $\tan x$
2. The sine and cosine rules for solving problems with triangles
3. How to calculate the area of a sector of a circle and the length of an arc of a circle
4. How to use trigonometry in 3D
5. How to draw graphs of $\sin x$, $\cos x$, $\tan x$, $a \sin n\theta$ and $a \cos n\theta$ for $a, n \in N$
6. How to solve equations of the form $\sin n\theta = 0$ or $\cos n\theta = \frac{1}{2}$ giving *all* solutions.
7. The derivation and use of various trigonometric formulae.

Definitions

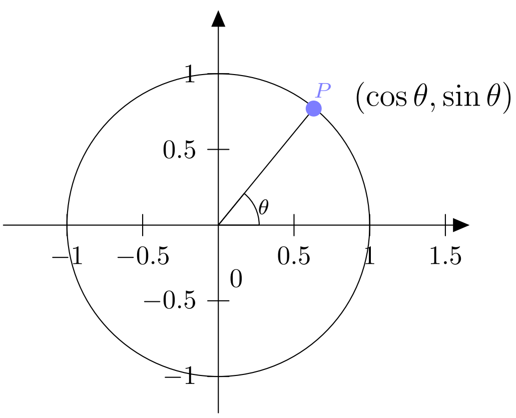


$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

The functions can also be defined on the unit circle



with

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Note that $\cos\theta$ is the x-coordinate of a point on the unit circle while $\sin\theta$ is the y-coordinate of the same point. The definition on the circle allows you to see immediately that

$$\sin^2\theta + \cos^2\theta = 1$$

since a unit circle, by definition, has a radius of 1.

Armed with these definitions you will find that you can solve quite a lot of the type of problems asked in the Leaving Certificate.

Example — Leaving Certificate 2011 Q8(a)(i)

A tower that is part of a hotel has a square base of side 4 metres and a roof in the form of a pyramid. The owners plan to cover the roof with copper. To find the amount of copper needed, they need to know the total area of the roof.

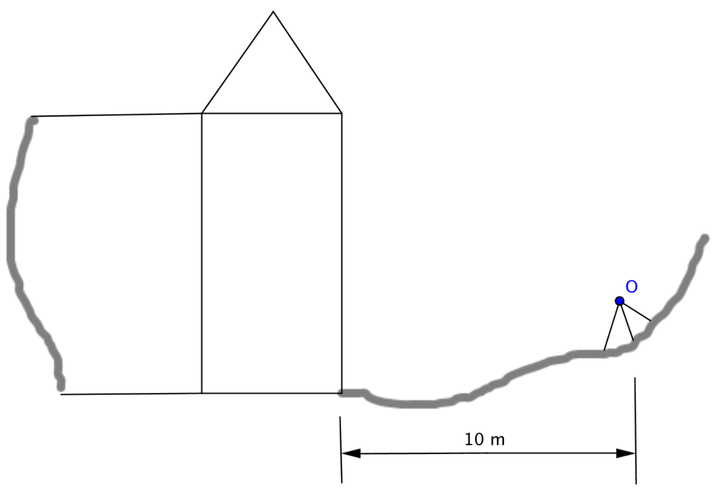
A surveyor stands 10 metres from the tower, measured horizontally, and makes observations of angles of elevation from the point O as follows:

The angle of elevation of the top of the roof is $46°$.

The angle of elevation of the closest point at the bottom of the roof is $42°$.

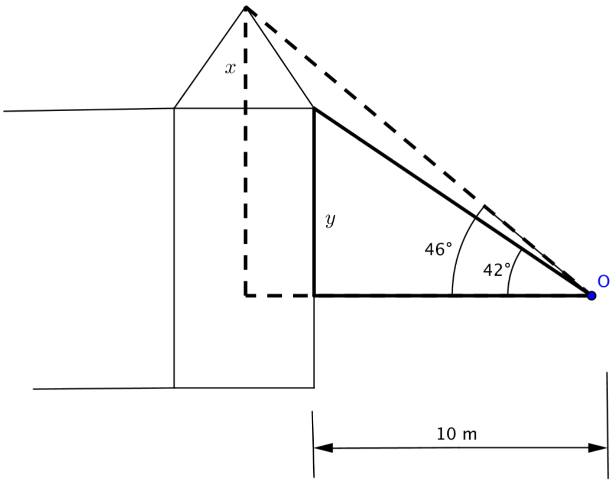
The angle of depression of the closest point at the bottom of the tower is $9°$.

Find the vertical height of the roof.



Solution

To solve a problem like this you must first extract from the problem the important details. The next diagram shows the problem broken into a series of line segments that make up a triangle:



This allows us to state that, in the solid triangle

$$\begin{aligned} \tan 42^\circ &= \frac{y}{10} \\ y &= 10 \tan 42^\circ \\ \Rightarrow y &= 9.004 \end{aligned}$$

While in the dashed triangle

$$\begin{aligned} \tan 46^\circ &= \frac{x+y}{12} \\ x+y &= 12 \tan 46^\circ \\ \Rightarrow x &= 12.426 - 9.004 = 3.42\,\text{m} \end{aligned}$$

Sine and Cosine rules

The Sine rule tells us that in any triangle if we label the sides opposite the angles A, B and C as a, b and c then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

while the Cosine rule states, using the same labeling, that

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Example — Sample Paper

A stand is being used to prop up a portable solar panel. It consists of a support that is hinged to the panel near the top, and an adjustable strap joining the panel to the support near the bottom.

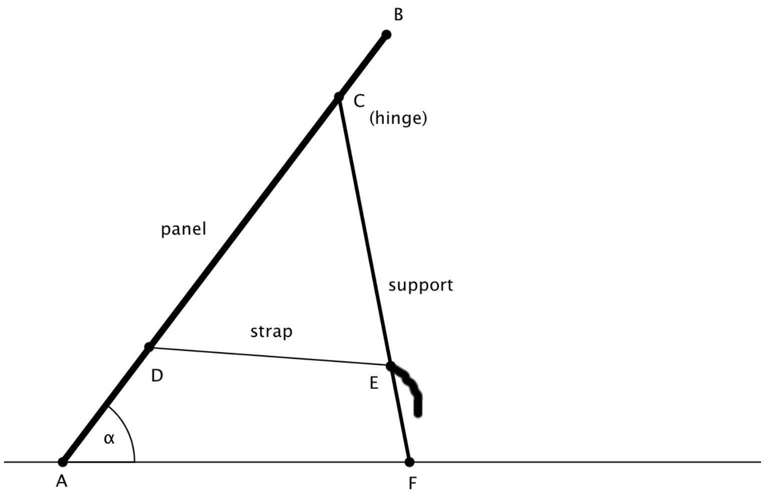
By adjusting the length of the strap, the angle between the panel and the ground can be changed.

The dimensions are as follows:

- $|AB| = 30$ cm

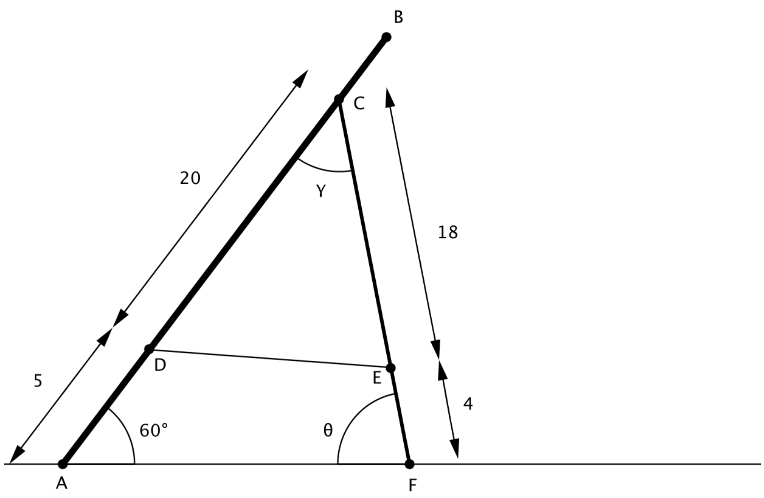
- $|AD| = |CB| = 5$ cm
- $|CF| = 22$ cm
- $|EF| = 4$ cm

Find the length of the strap $[DE]$ such that the angle α between the panel and the ground is $60°$



Solution

As before, break the figure down into triangles as follows:



In the triangle ABF we have, using the Sine rule,

$$\begin{aligned}
\frac{\sin 60°}{22} &= \frac{\sin\theta}{25} \\
\Rightarrow \sin\theta &= \frac{25\sin 60°}{22} \\
&= 0.9841 \\
\Rightarrow \theta &= 79.77°
\end{aligned}$$

Because the three angles in a triangle add up to $180°$ we have

$$\gamma = 180° - 139.77° = 40.23°$$

Now use the Cosine rule to get

$$\begin{aligned} |DE|^2 &= 20^2 + 18^2 - 2 \times 20 \times 18 \cos 40.23° \\ &= 174.31 \\ \Rightarrow |DE| &= 13.2\,\text{cm} \end{aligned}$$

Sectors and Arcs

Remember that the length, s, of an arc of a circle of radius r that subtends an angle of θ radians is given by

$$s = r\theta$$

while the area of the sector is

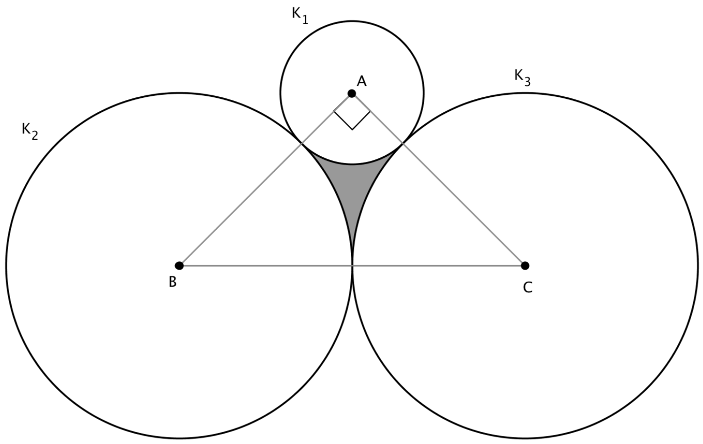
$$A = \frac{1}{2}r^2\theta$$

Example — Leaving Cert 2004 4(c)

A, B and C are the centers of circles K_1, K_2 and K_3 respectively. The three circles touch externally and $AB \perp AC$. K_2 and K_3 each have a radius of $2\sqrt{2}\,\text{cm}$.

(a) Find, in surd form, the length of the radios of K_1.

(b) Find the area of the shaded region in terms of π.



Solution

For the first part we use the fact that $|\angle ABC| = 45^\circ$ so

$$\begin{aligned} \cos 45^\circ &= \frac{|AB|}{|BC|} \\ &= \frac{|AB|}{4\sqrt{2}} \\ \Rightarrow |AB| &= 4\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) \\ &= 4 \\ \Rightarrow \text{ radius of } K_1 &= 4 - 2\sqrt{2} \end{aligned}$$

For the second part we will need to calculate areas of sectors.
Area of shaded region = Area of triangle $ABC - K_1$ sector $-K_2$ sector $-K_3$ sector.
Area of triangle $ABC = \frac{1}{2} \times 4 \times 4 = 8$.
Let the area of the smaller K_1 sector be A_1 then

$$\begin{aligned} A_1 &= \frac{1}{2}r^2\theta \\ &= \frac{1}{2}\left(4 - 2\sqrt{2}\right)^2 \frac{\pi}{2} \\ &= \frac{1}{4}\pi\left(16 - 16\sqrt{2} + 8\right) \\ &= 6\pi - 4\sqrt{2}\pi \end{aligned}$$

If we let A_2 be the area of the smaller sector in circle K_2 then A_2 is also the area of the smaller sector in K_3. This gives us

$$\begin{aligned} 2A_2 &= 2 \times \frac{1}{2}\left(2\sqrt{2}\right)^2 \frac{\pi}{4} \\ &= 2\pi \end{aligned}$$

Combining these three results we get

$$\begin{aligned} \text{Area of the shaded region} &= 8 - 6\pi + 4\sqrt{2}\pi - 2\pi \\ &= 8 - 8\pi + 4\sqrt{2}\pi \end{aligned}$$

Trigonometry in 3D

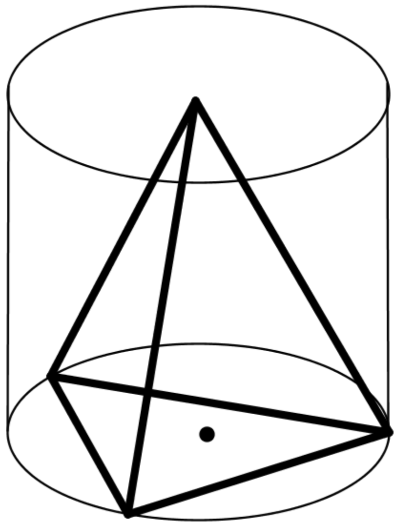
The trick here is to break down the problem into different 2D portions. Then your knowledge of trigonometry will help you do the problem.

Example — Sample Paper 2011

A regular tetrahedron has four faces, each of which is an equilateral triangle.

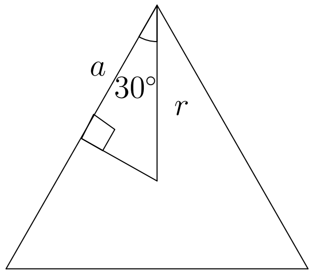
A wooden puzzle consists of several pieces that can be assembled to make a regular tetrahedron. The manufacturer wants to package the assembled tetrahedron in a clear cylindrical container, with one face flat against the bottom.

If the length of one edge of the tetrahedron is $2a$, show that the volume of the smallest possible cylindrical container is $\left(\frac{8\sqrt{6}}{9}\right)\pi a^3$.



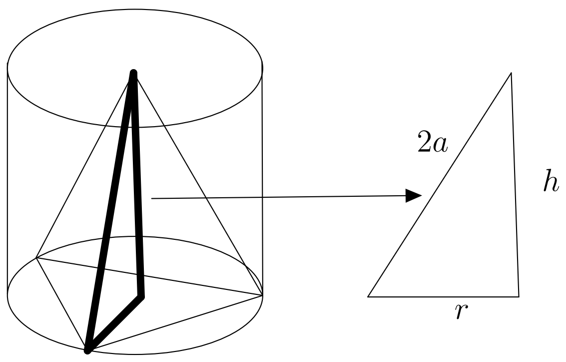
Solution

First consider the equilateral triangle that forms the base of the tetrahedron. Geometry comes in here immediately. The circle that forms the base of the cylinder is the circumcircle of the triangle. Its centre will be where the perpendicular bisectors of the sides meet. Since the triangle is equilateral this point coincides with the incentre, where the bisectors of the angles meet. Hence, if r is the radius of the base of the cylinder and A is its area



$$\begin{aligned} \cos 30^\circ &= \frac{a}{r} \\ &= \frac{\sqrt{3}}{2} \\ \Rightarrow r &= \frac{2a}{\sqrt{3}} \\ A &= \pi r^2 \\ &= \frac{4\pi a^2}{3} \end{aligned}$$

Now look at the tetrahedron and the solid triangle drawn in the next diagram.



You can see that

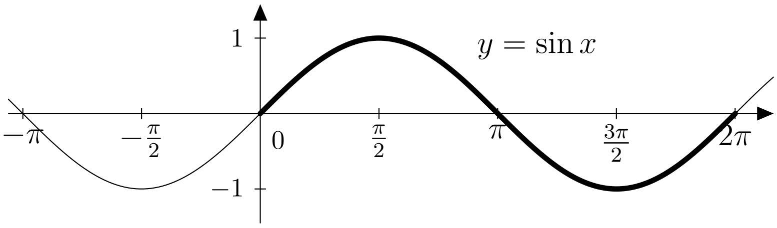
$$\begin{aligned} h^2 &= 4a^2 - r^2 \\ &= 4a^2 - \frac{4a^2}{3} \\ &= \frac{8a^2}{3} \\ \Rightarrow h &= \frac{\sqrt{8}a}{\sqrt{3}} \end{aligned}$$

We use the values derived for h and r in the formula for the volume of a cylinder to get

$$\begin{aligned} V &= \pi r^2 h \\ &= \frac{4\pi a^2}{3} \times \frac{\sqrt{8}a}{\sqrt{3}} \\ &= \frac{4\pi 2\sqrt{2}a^3}{3\sqrt{3}} \\ &= \frac{8\pi\sqrt{2}\sqrt{3}a^3}{3\sqrt{3}\sqrt{3}} \\ &= \left(\frac{8\sqrt{6}}{9}\right)\pi a^3 \end{aligned}$$

Graph Drawing

You *must* know what the graphs of the three trigonometric functions look like. Otherwise you will not be able to draw curves of the form $a \sin n\theta$ or $a \cos n\theta$. The fundamental graphs are shown below.



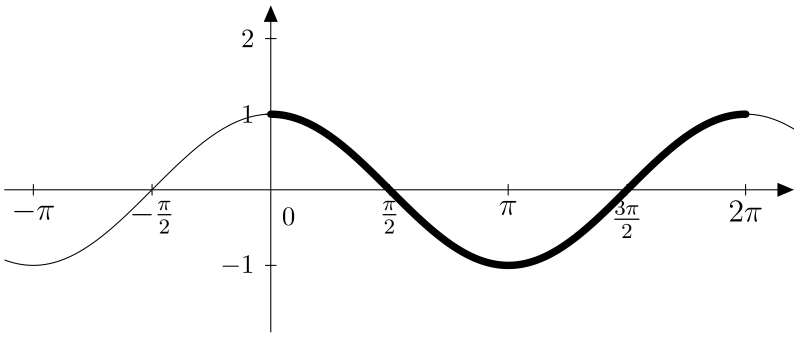
The heavy line shows one period of the graph. The period is the horizontal distance covered before the graph repeats. The period of $\sin x$ is 2π. The range is the interval from the lowest point to the highest. For $\sin x$ this will be $[-1, 1]$. It is important to be aware of the following values:

$$\begin{aligned}\sin 0 &= 0\\ \sin\frac{\pi}{2} &= 1\\ \sin\pi &= 0\\ \sin\frac{3\pi}{2} &= -1\\ \sin 2\pi &= 0\end{aligned}$$

Since the range of $\sin x$ is $[-1, 1]$ it follows that the range of $a\sin x$ must be $[-a, a]$.
The period is not affected by multiplying the function $\sin x$. It *is* affected by multiplying the value of x. If you draw the graph of $\sin 2x$ you will be getting the value of $\sin 2\pi$ when x is only π. This means that the graph will start to repeat after π units instead of 2π units. In general, the function

$$f(x) = a\sin nx$$

has period $\frac{2\pi}{n}$ and range $[-a, a]$.
Similarly the graph of $\cos x$ looks like the diagram below.



The comments about the period and range apply to the graphs of $\cos x$ and $a\cos nx$ also. Be aware that

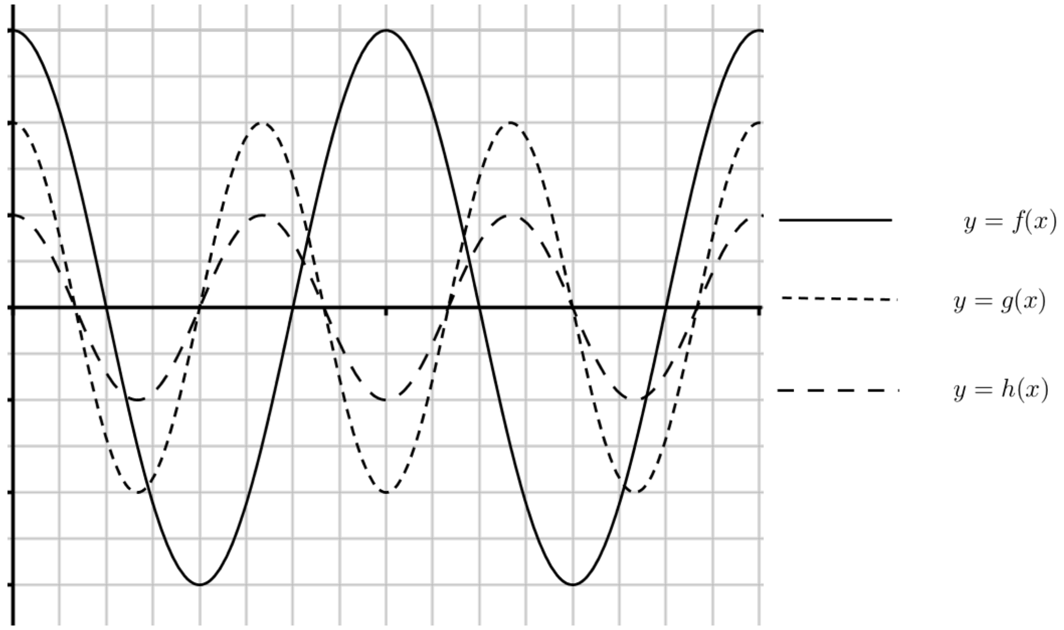
$$\begin{aligned}\cos 0 &= 1\\ \cos\frac{\pi}{2} &= 0\\ \cos\pi &= -1\\ \cos\frac{3\pi}{2} &= 0\\ \cos 2\pi &= 1\end{aligned}$$

Example — Leaving Certificate 2010 Q5(b)

The graphs of three functions are shown on the diagram below. The scales on the axes are not labelled. The three functions are

$$\begin{aligned}x &\to \cos 3x\\ x &\to 2\cos 3x\\ x &\to 3\cos 2x\end{aligned}$$

Identify which function is which, and write your answers in the spaces below the diagram.



$f : x \to$ $\qquad$ $g : x \to$ $\qquad$ $h : x \to$

Label the scales on the axes in the diagram.

Solving equations

Trigonometric equations have more than one solution. In the diagram you can see that the horizontal and vertical lines each cut the unit circle twice. A horizontal line corresponds to an equation of the form

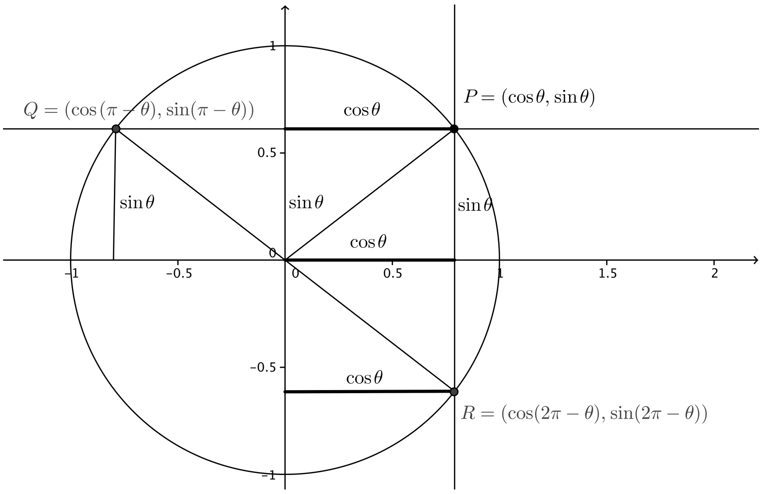
$$\sin n\theta = k$$

since the y-coordinates of the points where the line cuts the circle will be equal.

Similarly, a vertical line cutting the circle is equivalent to an equation of the form

$$\cos n\theta = k$$

since this time the x-coordinates of the points of intersection will be equal.



As well as this, Everytime you go around the circle you add 2π to the angle and you end up at the same place. This means that whenever you find a solution to a trigonometric equation you can add, or subtract, any whole number multiples of 2π and you will get another solution to the equation.

Example — Leaving Certificate 2010 Q5(a)

Solve the equation $\cos 3\theta = \frac{1}{2}$, for $\theta \in R$, (where θ is in radians).

Remembering that

$$\cos(2\pi - \theta) = \cos\theta$$

we get the specific solutions

$$\cos 3\theta = \frac{1}{2} \Rightarrow 3\theta = \frac{\pi}{3} \text{ or } 3\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

We can get general solutions by adding any integral multiple of 2π. Hence

$$\begin{aligned} 3\theta &= \frac{\pi}{3} + 2n\pi, n \in Z \\ \theta &= \frac{\pi}{9} + \frac{2n\pi}{3} \\ &= \frac{6n\pi + \pi}{9} \end{aligned}$$

and also

$$\begin{aligned} 3\theta &= \frac{5\pi}{3} + 2n\pi, n \in Z \\ \theta &= \frac{5\pi}{9} + \frac{2n\pi}{3} \\ &= \frac{6n\pi + 5\pi}{9} \end{aligned}$$