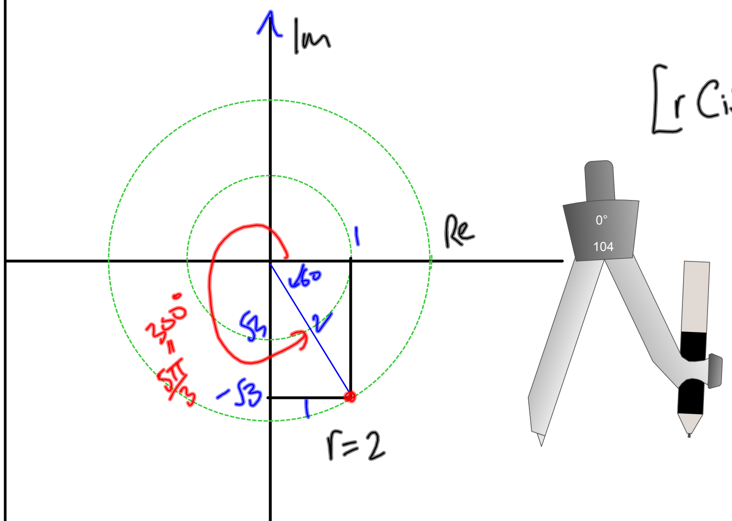
Complex No.s

$z = 1 - \sqrt{3}i$ $\quad z^4 = ?$

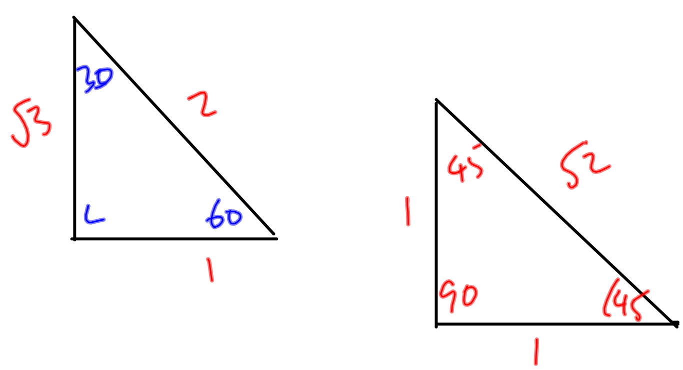
$[r \, CiS\theta]^n = r^n \, CiS \, n\theta$




$$z^4 = \left[2 \, CiS \, \frac{5\pi}{3}\right]^4 = 2^4\left[\cos 4\left(\frac{5\pi}{3}\right) + i\sin 4\left(\frac{5\pi}{3}\right)\right]$$

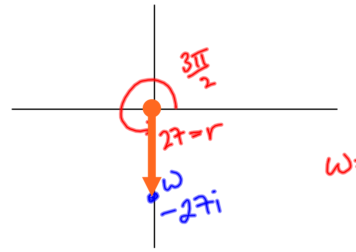
$$= -8 + 8\sqrt{3}i$$

Special triangles



Solve $z^3 = -27i$ $\Rightarrow$ $z = (-27i)^{\frac{1}{3}}$ ($-27i = \omega$)

Sketch




$\omega = 27 \text{ Cis } \frac{3\pi}{2}$

z? 3 solutions

$\Rightarrow$ 3 Arguments ① $\frac{3\pi}{2}$ ② $\frac{3\pi}{2} + 2\pi = \frac{7\pi}{2}$ ③ $\frac{7\pi}{2} + 2\pi = \frac{11\pi}{2}$

Modulus $z = 3$

① $z_1 = 3 \text{ Cis } \frac{3\pi}{2} = 3i$

② $z_2 = 3 \text{ Cis } \frac{7\pi}{2} = -3i$

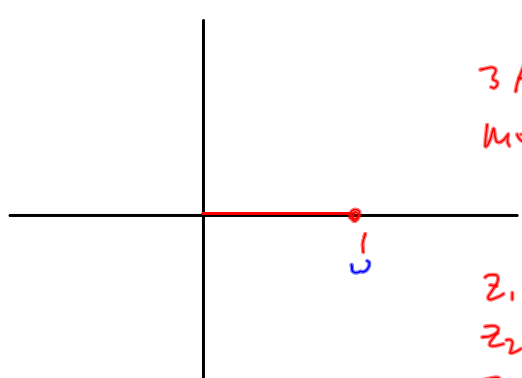
③ $z_3 = 3 \text{ Cis } \frac{11\pi}{2} = -3i$

Check : is $(3i)^3 = 27(-i) = -27i$ ✓ $-27i$

# Find cubed roots of unity?

$z^3 = 1$ $z = ?$

$z = (1)^{\frac{1}{3}}$ ($1 = \omega$)




3 Arguments = 0 , $2\pi$ , $4\pi$

Modulus = 1

$z_1 = \text{Cis } 0$

$z_2 = \text{Cis } 2\pi$

$z_3 = \text{Cis } 4\pi$

## Sequence & series

$1.\dot{4}\dot{7}$ as geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r}$$

$$1 + \frac{47}{100} + \frac{47}{10{,}000} + \frac{47}{1{,}000{,}000} + \ldots$$

$\times r$

$1 + S_\infty$

$a = \frac{47}{100}$ $\quad r = \frac{1}{100}$

$$1.\dot{4}\dot{7} = 1 + \frac{\frac{47}{100}}{1-\frac{1}{100}} = 1 + \frac{47}{99} = \frac{146}{99}$$

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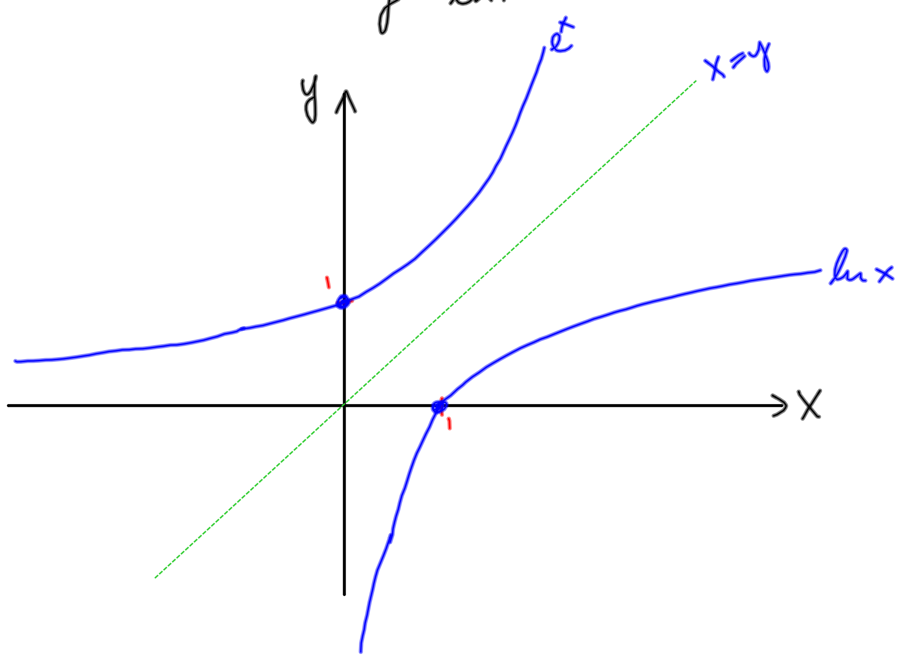
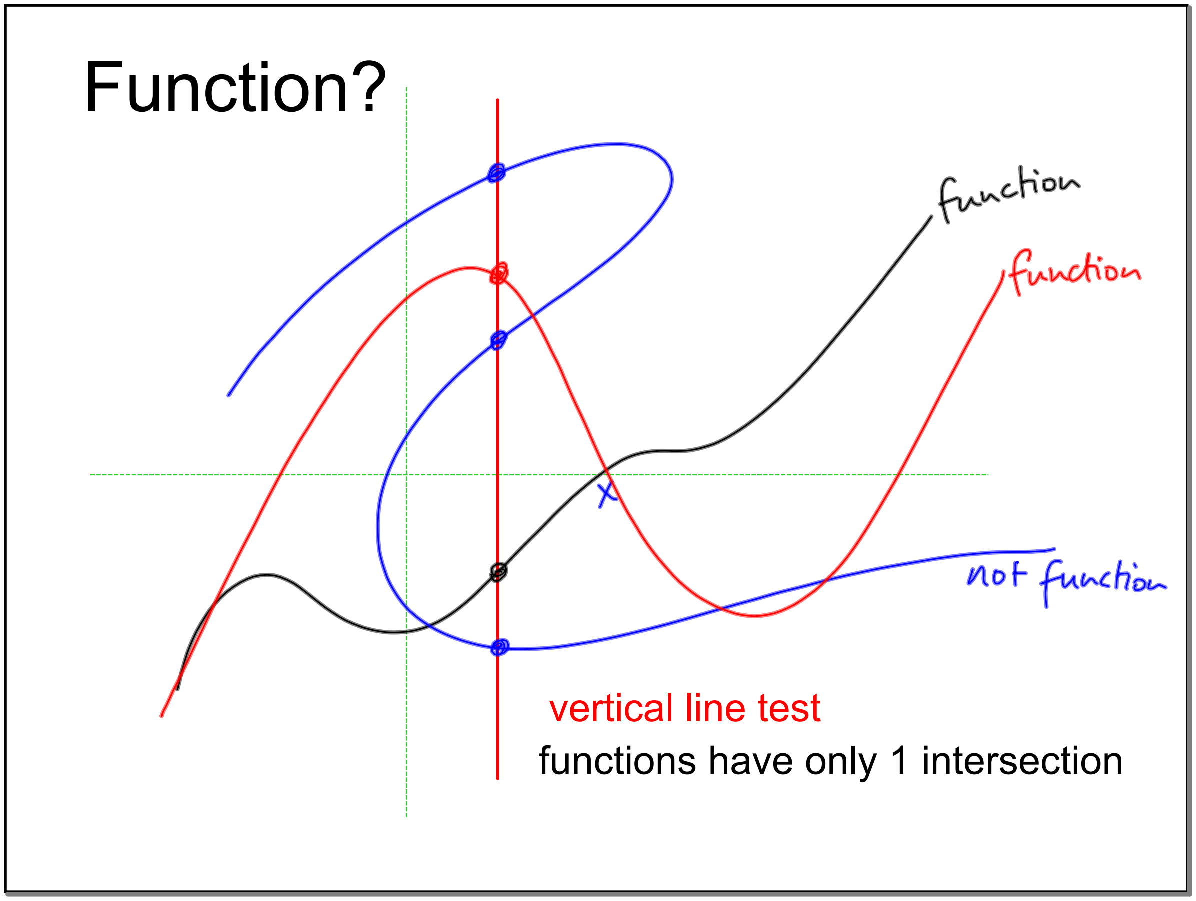
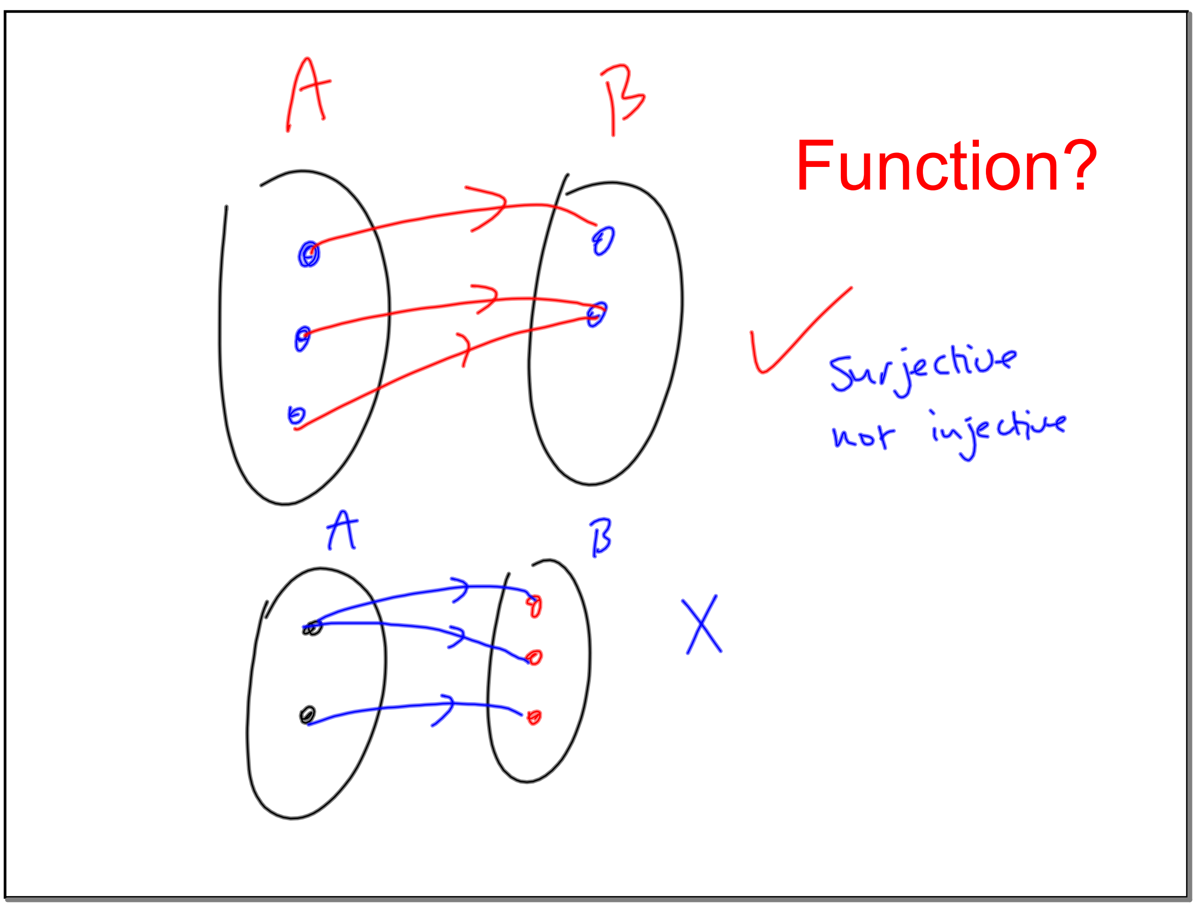
sketch $y = e^x$

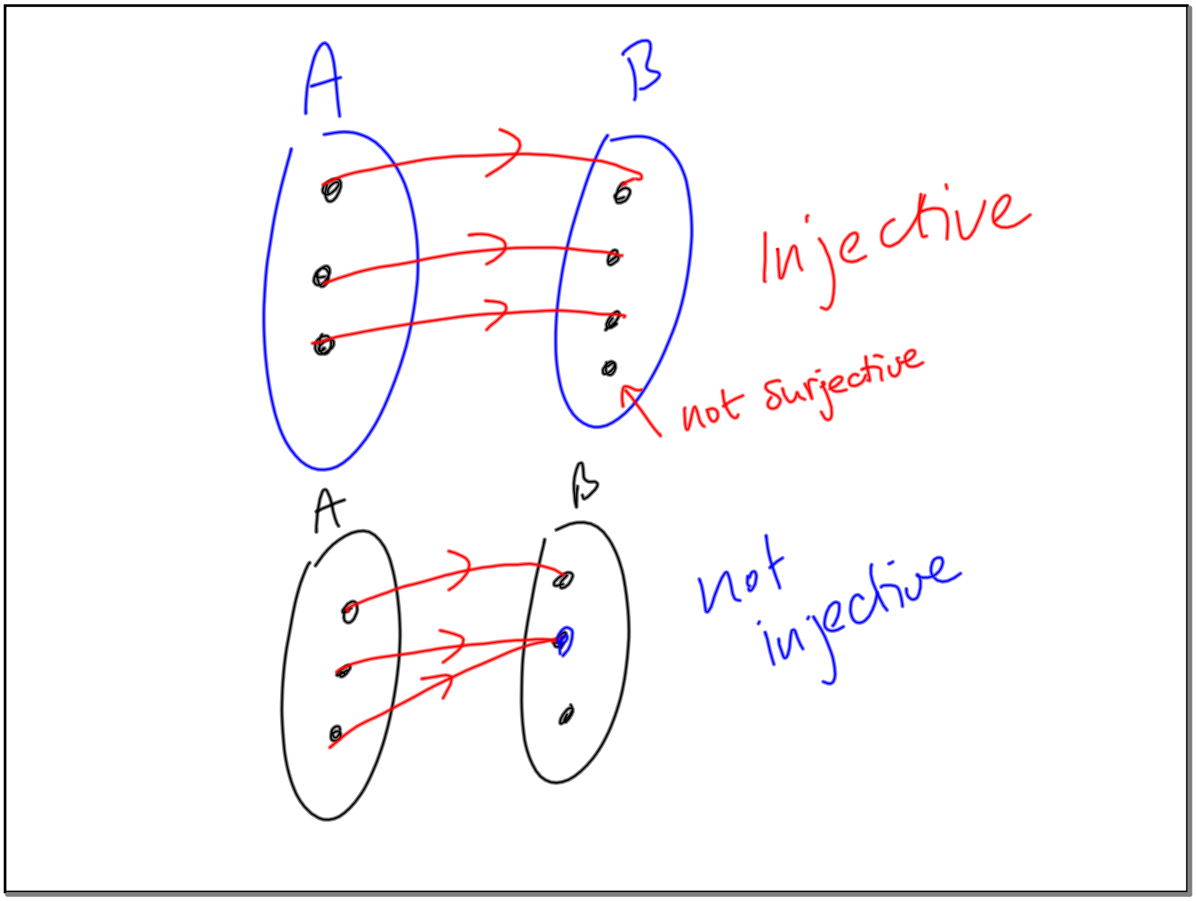
$y = \ln x$

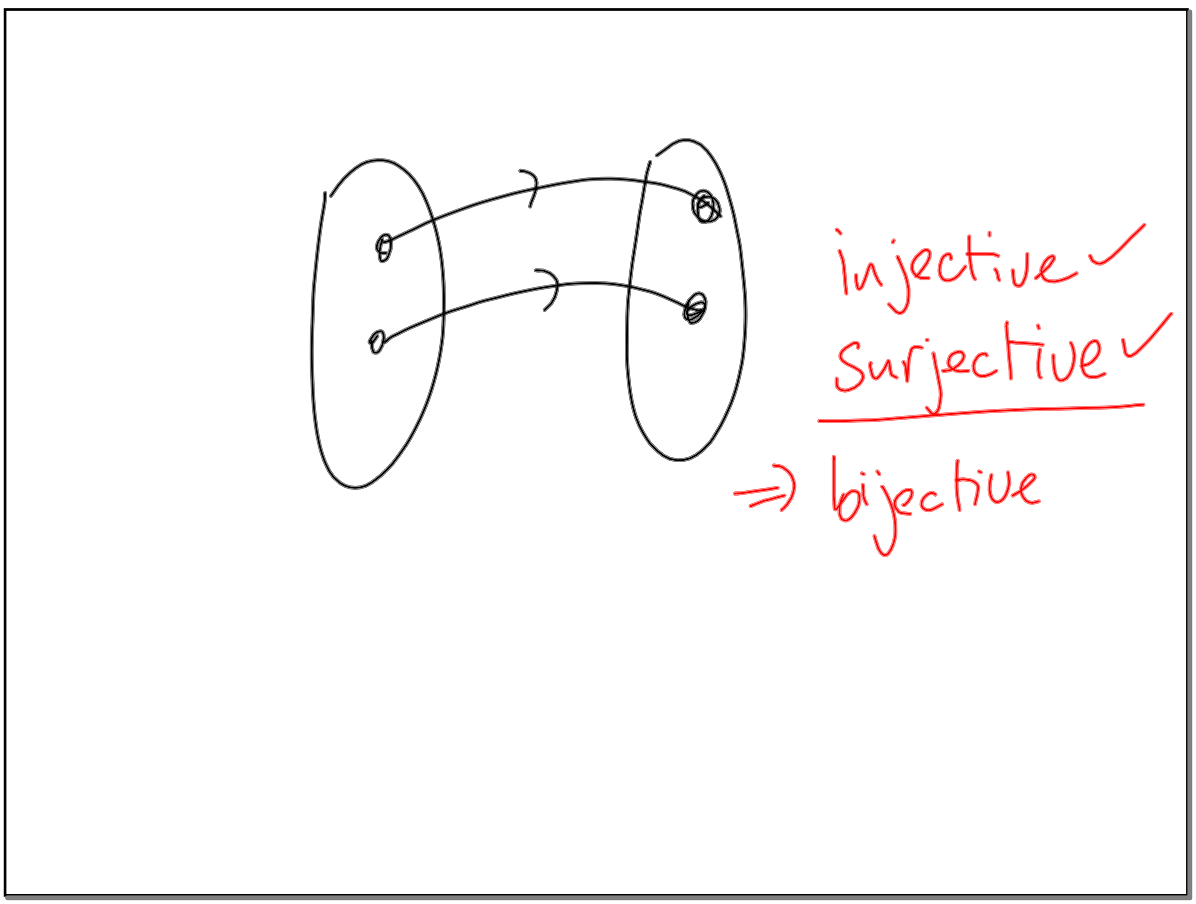
INVERSE FUNCTIONS




Function?
function
function
not function
x
vertical line test
functions have only 1 intersection


A
B
Function?
Surjective
not injective
A
B
X


A
B
Injective
not surjective
A
B
not
injective


injective ✓
surjective ✓
⇒ bijective

**4. Inverse functions**

e.g. $f$ is the function $f : x \to \frac{5x-2}{2x+3}$.

① $y = ?$

② $x = ?$

③ replace

$x \to f^{-1}(x)$

$y \to x$

$y = \frac{5x-2}{2x+3}$

$y(2x+3) = 5x-2$

$2xy + 3y = 5x-2$

$3y+2 = 5x - 2xy$

$3y+2 = x(5-2y)$

$x = \frac{3y+2}{5-2y}$

$f^{-1}(x) = \frac{3x+2}{5-2x}$

Differentiate from 1st Principles

$y = 2x^2 + x - 1$

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

- $f(x) = 2x^2 + x - 1$
- $f(x+h) = 2(x+h)^2 + (x+h) - 1$
  $= 2[x^2 + 2xh + h^2] + x + h - 1$
  $= 2x^2 + 4xh + 2h^2 + x + h - 1$
- $\frac{f(x+h) - f(x)}{h} = \frac{4xh + 2h^2 + h}{h}$
  $= 4x + 2h + 1$

$f'(x) = 4x + 2(0) + 1$

$f'(x) = 4x + 1$

## Differentiation by rule

- differentiate the following functions
  - polynomial
  - exponential
  - trigonometric
  - rational powers
  - inverse functions
  - logarithms
- find the derivatives of sums, differences, products, quotients and compositions of functions of the above form

Chain Rule

**Quotient rule**

$$y=\frac{u}{v} \Rightarrow \frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

Chain rule

product rule

$$y = xe^{2x}$$

u = $x$, v = $e^{2x}$

| $f(x)$ | $\to$ $f'(x)$ |
|---|---|
| $e^x$ | $e^x$ |
| $e^{ax}$ | $ae^{ax}$ |

**Product rule**

$$y=uv \Rightarrow \frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$$

$u = x$ $\quad$ $v = e^{2x}$

$\frac{du}{dx} = 1$ $\quad$ $\frac{dv}{dx} = 2e^{2x}$

$$\frac{dy}{dx} = x(2e^{2x}) + e^{2x}(1)$$

$$= 2xe^{2x} + e^{2x}$$

OR $= e^{2x}(2x+1)$

Chain Rule example

$$y = \cos^2(3x^2+1)$$

$$= [\cos(3x^2+1)]^2$$

$$\frac{dy}{dx} = 2[\cos(3x^2+1)]^1 . [-\sin(3x^2+1)] . [6x]$$

Diff of outside × Diff. of middle × diff. of inside

$$= -12x\cos(3x^2+1)\sin(3x^2+1)$$

(i) $y = \tan^{-1}\left(\frac{x}{\sqrt{9-x^2}}\right)$  $\frac{dy}{dx} = ?$  let $X = \frac{x}{\sqrt{9-x^2}}$

$f(x) \rightarrow f'(x)$

| | |
|---|---|
| $\tan^{-1}\frac{x}{a}$ | $\frac{a}{a^2+x^2}$ |

$y = \tan^{-1}\frac{X}{1}$ ← a = 1

$\frac{dy}{dx} = \left(\frac{1}{1^2 + X^2}\right)\left(\text{diff. of inside}\right)$

Diff. of inside?

$\frac{dX}{dx} = ?$ Quotient

diff inside

$u = x$  $v = \sqrt{9-x^2} = (9-x^2)^{\frac{1}{2}}$

$\frac{du}{dx} = 1$  $\frac{dv}{dx} = \frac{1}{2}(9-x^2)^{-\frac{1}{2}} \cdot (-2x)$

$= -x(9-x^2)^{-\frac{1}{2}}$

**Quotient rule**

$$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dX}{dx} = \frac{\sqrt{9-x^2}(1) - (x)(-x(9-x^2)^{-\frac{1}{2}})}{9-x^2}$$

$$= \frac{\frac{\sqrt{9-x^2}}{\sqrt{9-x^2}} - \frac{x^2}{\sqrt{9-x^2}\sqrt{9-x^2}}}{\frac{9-x^2}{\sqrt{9-x^2}}}$$

$\div \sqrt{9-x^2}$ above and below

$$= \frac{1 - \frac{x^2}{9-x^2}}{\sqrt{9-x^2}}$$

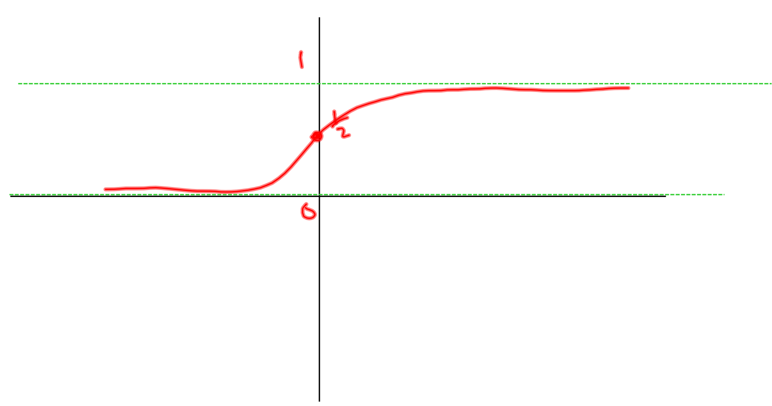
**4. Curve sketching**

e.g. $f : x \rightarrow \frac{e^x}{e^x+1}$ is a function defined for all $x \in \mathbb{R}$.

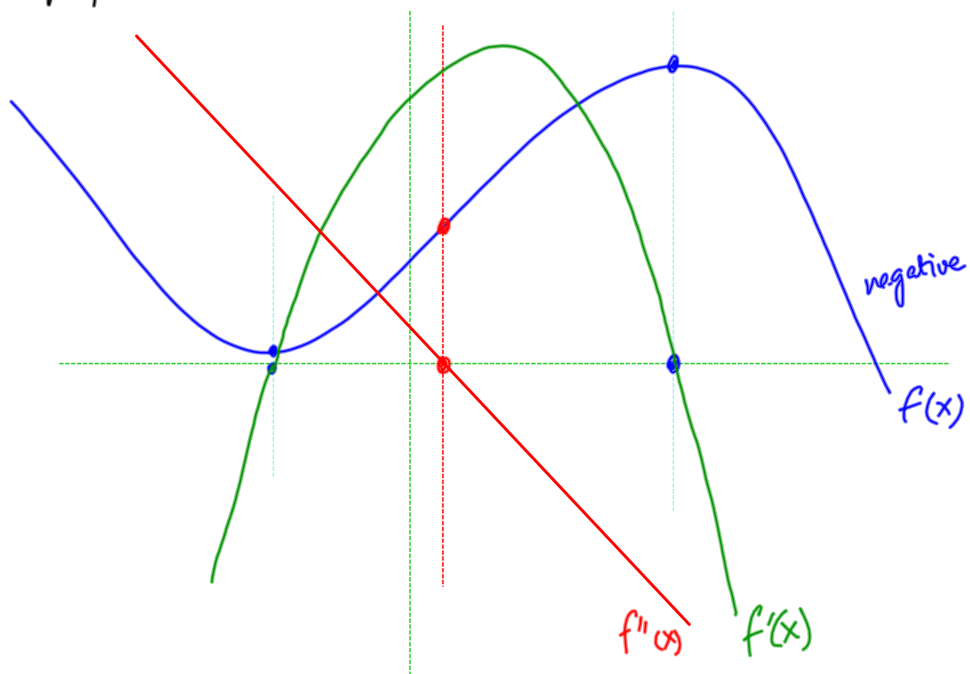
(i) Find the horizontal asymptotes of the curve $y = f(x)$.

if no turning pts $\Rightarrow \frac{dy}{dx} \neq 0$

$\frac{e^{99}}{e^{99}+1} \approx 1$  $\frac{e^{-999}}{e^{-999}+1} \approx 0$



## Slope functions




## Integration

(1) $\int 2x^3 . dx = \frac{2x^4}{4} + c = \frac{x^4}{2} + c$

(2) $\int 3^x . dx = \frac{3^x}{\ln 3} + c$

| $f'(x)$ → | $f(x)$ |
|---|---|
| $a^x$ $(a > 0)$ | $\frac{a^x}{\ln a}$ |

(3) $\int 2 \sin 3x . dx = -2 \cos 3x \left[\frac{1}{3}\right] + c$

$= -\frac{2}{3} \cos 3x + c$

| $f'(x)$ → | $f(x)$ |
|---|---|
| $\cos x$ | $\sin x$ |
| $\sin x$ | $-\cos x$ |

* $f'(x)$ → $f(x)$

$\cos ax$ → $\frac{1}{a} \sin ax$

$\sin ax$ → $-\frac{1}{a} \cos ax$

$$y = -\cos 3x$$

$$\frac{dy}{dx} = -(-\sin 3x)\cdot(3)$$

$$= +3\sin 3x$$

×3

÷3

* NB Learn

$$\int \sin 3x \cdot dx$$

$$= -\frac{\cos 3x}{3} + c$$

Dealing With constant multipliers during integration:

you can consider them in front of the integral.

$$\int 3x^3\,dx$$

$$= 3\int x^3\,dx$$

$$= 3\left(\frac{x^4}{4}\right) + c$$

$$a(x-b)^2 + c$$

max/min = $(b, c)$


**7. Solving quadratic equations**

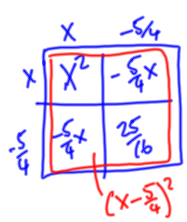
e.g. solve the equation

$$2x^2 - 5x - 3 = 0$$

(i) by factors,
(ii) by completing the square,
(iii) by the quadratic formula

Completing the square

$(a-b)^2 = a^2 - 2ab + b^2$




$$2\left[x^2 - \frac{5}{2}x - \frac{3}{2}\right] = 0$$

$$= 2\left[x^2 - \frac{5}{2}x + \frac{25}{16} - \frac{25}{16} - \frac{3}{2}\right]$$

$$= 2\left[(x - 5/4)^2 - 25/16 - 3/2\right]$$

$$= 2\left[\left(x - \frac{5}{4}\right)^2 - \frac{25}{16} - \frac{3}{2}\right]$$

$$= 2\left[\left(x - \frac{5}{4}\right)^2 - \frac{49}{16}\right]$$

$$= 2\left(x - \frac{5}{4}\right)^2 - \frac{49}{8}$$

vertex $\left(\frac{5}{4}, \frac{49}{8}\right)$

Solve ?

$$2\left(x - \frac{5}{4}\right)^2 - \frac{49}{8} = 0$$

$$\left(x - \frac{5}{4}\right)^2 = \frac{49}{8}\left(\frac{1}{2}\right)$$

$$x - \frac{5}{4} = \pm\sqrt{\frac{49}{16}}$$

$$x = \pm\frac{9}{4} \cdot - \frac{5}{4}$$

$$\Rightarrow x = \frac{4}{4} = 1 \quad \text{or} \quad x = -\frac{14}{4} = -\frac{7}{2}$$