Differentiation by Rule (HL)

Using Product, Quotient & Chain Rules

Exercise 6D

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P.190

Q1 (i) $y = (3x+4)(x-2)$

$u = 3x+4$ $\quad v = x-2$

$\frac{du}{dx} = 3$ $\quad \frac{dv}{dx} = 1$

$$\frac{dy}{dx} = (3x+4)(1) + (x-2)(3)$$

$$= 3x + 4 + 3x - 6$$

$$= 6x + 2$$

Product Rule

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Q1
(ii) $y = (3x-4)(4x+5)$

$u = 3x-4$ $\quad v = 4x+5$

$\frac{du}{dx} = 3$ $\quad \frac{dv}{dx} = 4$

Product Rule

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = (3x-4)(4) + (4x+5)(3)$$
$$= 12x - 16 + 12x + 15$$
$$= 24x - 1$$

Q1 (iii) $y = (x^2+2)(x-1)$

$u = x^2+2$ $\quad v = x-1$

$\frac{du}{dx} = 2x$ $\quad \frac{dv}{dx} = 1$

Product Rule

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = (x^2+2)(1) + (x-1)(2x)$$
$$= x^2 + 2 + 2x^2 - 2x$$
$$= -x^2 - 2x + 2$$

Product Rule

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

(iv) $y = (2x+1)(x^2-2)$

$u = 2x+1$ $\quad v = x^2-2$

$\frac{du}{dx} = 2$ $\quad \frac{dv}{dx} = 2x$

$$\frac{dy}{dx} = (2x+1)(2x) + (x^2-2)(2)$$
$$= 4x^2 + 2x + 2x^2 - 4$$
$$= 6x^2 + 2x - 4$$

(v)

Product Rule

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$y = (1-x)(2-x^2)$

$u = 1-x$ $\quad v = 2-x^2$

$\frac{du}{dx} = -1$ $\quad \frac{dv}{dx} = -2x$

$$\frac{dy}{dx} = (1-x)(-2x) + (2-x^2)(-1)$$
$$= -2x + 2x^2 - 4 + x^2$$
$$= 3x^2 - 2x - 4$$

(v.)

$$y = (x^3 - 1)(2x + 1)$$

Product Rule

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$u = x^3 - 1$ $\quad v = 2x + 1$

$\frac{du}{dx} = 3x^2$ $\quad \frac{dv}{dx} = 2$

$$\frac{dy}{dx} = (x^3 - 1)(2) + (2x + 1)(3x^2)$$
$$= 2x^3 - 2 + 6x^3 + 3x^2$$
$$= 8x^3 + 3x^2 - 2$$

Q2

(i) $f(x) = \frac{3x}{2x+6}$

QUOTIENT RULE

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$u = 3x$ $\quad v = 2x + 6$

$\frac{du}{dx} = 3$ $\quad \frac{dv}{dx} = 2$

$$f'(x) = \frac{(2x+6)(3) - (3x)(2)}{(2x+6)^2} = \frac{6x + 18 - 6x}{(2x+6)^2}$$

$$= \frac{18}{(2x+6)^2}$$

Q2 (ii)

$$f(x) = \frac{2x+3}{x-1}$$

$u = 2x+3$ $\quad v = x-1$

$\frac{du}{dx} = 2$ $\quad \frac{dv}{dx} = 1$

QUOTIENT RULE

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$f'(x) = \frac{(x-1)(2) - (2x+3)(1)}{(x-1)^2} = \frac{2x - 2 - 2x - 3}{(x-2)^2}$$

$$= \frac{-5}{(x-2)^2}$$

(iii)

$$f(x) = \frac{x^2}{2x+3}$$

$u = x^2$ $\quad v = 2x+3$

$\frac{du}{dx} = 2x$ $\quad \frac{dv}{dx} = 2$

QUOTIENT RULE

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$f'(x) = \frac{(2x+3)(2x) - (x^2)(2)}{(2x+3)^2} = \frac{4x^2 + 6x - 2x^2}{(2x+3)^2}$$

$$= \frac{2x^2 + 6x}{(2x+3)^2}$$

(iv) $f(x) = \dfrac{2x^2-1}{2x-3}$

QUOTIENT RULE

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$u = 2x^2-1$ $\quad v = 2x-3$

$\dfrac{du}{dx} = 4x$ $\quad \dfrac{dv}{dx} = 2$

$$f'(x) = \frac{(2x-3)(4x) - (2x^2-1)(2)}{(2x-3)^2} = \frac{8x^2-12x-4x^2+2}{(2x-3)^2}$$

(v) $f(x) = \dfrac{2x^3}{1-2x}$

QUOTIENT RULE

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$u = 2x^3$ $\quad v = 1-2x$

$\dfrac{du}{dx} = 6x^2$ $\quad \dfrac{dv}{dx} = -2$

$$\frac{dy}{dx} = \frac{(1-2x)(6x^2) - (2x^3)(-2)}{(1-2x)^2} = \frac{6x^2-12x^3+4x^3}{(1-2x)^2}$$

$$= \frac{6x^2-8x^3}{(1-2x)^2}$$

(vii) $f(x) = \dfrac{3x+2}{x^2-3}$

QUOTIENT RULE

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$u = 3x+2$ $\quad v = x^2-3$

$\dfrac{du}{dx} = 3$ $\quad \dfrac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{(x^2-3)(3) - (3x+2)(2x)}{(x^2-3)^2} = \frac{3x^2-9-6x^2-4x}{(x^2-3)^2}$$

$$= \frac{-3x^2-4x-9}{(x^2-3)^2}$$

Q3 $\quad y = \dfrac{x^2+1}{3x-1}$ $\quad$ find $\dfrac{dy}{dx}$ when $x=0$

QUOTIENT RULE

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$u = x^2+1$ $\quad v = 3x-1$

$\dfrac{du}{dx} = 2x$ $\quad \dfrac{dv}{dx} = 3$

$$\frac{dy}{dx} = \frac{(3x-1)(2x) - (x^2+1)(3)}{(3x-1)^2} = \frac{6x^2-2x-3x^2-3}{(3x-1)^2} = \frac{3x^2-2x-3}{(3x-1)^2}$$

$$\frac{dy}{dx}(x=0) = \frac{3(0)^2-2(0)-3}{(3(0)-1)^2} = -3$$

Q4 $y = \sqrt{x}(2x-1)$ $\frac{dy}{dx} = ?$ (express as single fraction)

$u = \sqrt{x} = x^{\frac{1}{2}}$ $\quad V = 2x-1$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ $\quad \frac{dv}{dx} = 2$

Product Rule

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = (\sqrt{x})(2) + (2x-1)\left(\frac{1}{2\sqrt{x}}\right) = \frac{2\sqrt{x}}{1} + \frac{2x-1}{2\sqrt{x}}$$

$$= \frac{(2\sqrt{x})(2\sqrt{x}) + (1)(2x-1)}{2\sqrt{x}} = \frac{4x+2x-1}{2\sqrt{x}} = \frac{6x-1}{2\sqrt{x}}$$

Q5 $y = (\sqrt{x}+4)(\sqrt{x}-4)$ Show $\frac{dy}{dx} = 1$

(use product rule! - note it would be easier to recognise difference of 2 squares here)

Product Rule

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$u = \sqrt{x}+4$ $\quad V = \sqrt{x}-4$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ $\quad \frac{dv}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$$\frac{dy}{dx} = (\sqrt{x}+4)\left(\frac{1}{2\sqrt{x}}\right) + (\sqrt{x}-4)\left(\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{1}{2} + \frac{2}{\sqrt{x}} + \frac{1}{2} - \frac{2}{\sqrt{x}} = 1$$

Q6 $y = \frac{x}{1-x^2}$ Show $\frac{dy}{dx} > 0$ for all $x \in \mathbb{R}$

$u = x$ $v = 1-x^2$

$\frac{du}{dx} = 1$ $\frac{dv}{dx} = -2x$

QUOTIENT RULE

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(1-x^2)(1) - (x)(-2x)}{(1-x^2)^2} = \frac{1-x^2+2x^2}{(1-x^2)^2} = \frac{1+x^2}{(1-x^2)^2}$$

Since $1+x^2$ and $(1-x^2)^2$ are both positive

$$\Rightarrow \frac{1+x^2}{(1-x^2)^2} > 0$$

Q7 CHAIN RULE

(i) $y = (x+4)^2$

$\frac{dy}{dx} = 2(x+4)(1)$

$= 2(x+4)$

(ii) $y = (2x-1)^3$

$\frac{dy}{dx} = 3(2x-1)^2(2)$

$= 6(2x-1)^2$

(iii) $y = (3x+5)^3$

$\frac{dy}{dx} = 3(3x+5)^2(3)$

$= 9(3x+5)^2$

(iv) $y = (x^2-1)^2$

$\frac{dy}{dx} = 2(x^2-1)(2x)$

$= 4x(x^2-1)$

Q7 CHAIN RULE

(v) $y = (2x^2+3)^4$

$\frac{dy}{dx} = 4(2x^2+3)^3(4x)$

$= 16x(2x^2+3)^3$

(vi) $y = (1-3x)^5$

$\frac{dy}{dx} = 5(1-3x)^4(-3)$

$= -15(1-3x)^4$

Q8

(i) $f(x) = \sqrt{4x+1} = (4x+1)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(4x+1)^{-\frac{1}{2}}(4) = 2(4x+1)^{-\frac{1}{2}}$

(ii) $f(x) = \sqrt{x^2-4} = (x^2-4)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(x^2-4)^{-\frac{1}{2}}(2x) = x(x^2-4)^{-\frac{1}{2}}$

(iii) $f(x) = \sqrt{x^3-2x} = (x^3-2x)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(x^3-2x)^{-\frac{1}{2}}(3x) = \frac{3}{2}x(x^3-2x)^{-\frac{1}{2}}$

Q9 (i) $y = 2x(2x+5)^3$ (PRODUCT WITH CHAIN RULE)

$u = 2x$ $\quad v = (2x+5)^3$

$\frac{du}{dx} = 2$ $\quad \frac{dv}{dx} = 3(2x+5)^2(2) = 6(2x+5)^2$

Product Rule

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = 2x(6(2x+5)^2) + (2x+5)^3(2)$$

$$= 12x(2x+5)^2 + 2(2x+5)^3 \quad \checkmark \text{ [you can stop here!]}$$

$$= (2x+5)^2(12x + 2(2x+5)) \quad \text{[since } (2x+5)^2 \text{ is a HCF...]}$$

$$= (2x+5)^2(12x + 4x + 10)$$

$$= (2x+5)^2(6x+10)$$

Q9 (ii) $y = (x^2-1)(3x+2)^2$ (PRODUCT WITH CHAIN RULE)

$u = x^2-1$ $\quad v = (3x+2)^2$

$\frac{du}{dx} = 2x$ $\quad \frac{dv}{dx} = 2(3x+2)(3)$

$= 6(3x+2)$

Product Rule

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = (x^2-1)(6(3x+2)) + (3x+2)^2(2x)$$

$$= 6(x^2-1)(3x+2) + 2x(3x+2)^2 \quad \text{[NOTE: you can stop here]}$$

$$= (6x^2-6)(3x+2) + (6x^2+4x)(3x+2) \quad \text{[Since } (3x+2) \text{ is HCF...]}$$

$$= (3x+2)(6x^2-6+6x^2+4x)$$

$$= (3x+2)(12x^2+4x-6) = 2(3x+2)(6x^2+2x-3)$$

Q9 (iii) $y = (x+4)^2(x-2)$ (Product with Chain Rule)

$u = (x+4)^2$

$\frac{du}{dx} = 2(x+4)(1) = 2(x+4)$

$v = (x-2)$

$\frac{dv}{dx} = 1$

Product Rule

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = (x+4)^2(1) + (x-2)(2(x+4))$$

$$= (x+4)^2 + 2(x-2)(x+4) \quad \text{[Since } (x+4) \text{ is the HCF]}$$

$$= (x+4)(x+4+2x-4) = (x+4)(3x)$$

Q10 $y = (x^2-3)^3$ find $\frac{dy}{dx}$ when $x=1$

$$\frac{dy}{dx} = 3(x^2-3)^2(2x) \quad \text{[Chain Rule]}$$

$$= 6x(x^2-3)^2$$

$$\frac{dy}{dx}(x=1) = 6(1)(1^2-3)^2 = 6(1-3)^2 = 6(2)^2 = 24$$

Q4 $y = \dfrac{(2x-1)^2}{3x+4}$ find $\dfrac{dy}{dx}$ when $x=0$ [QUOTIENT AND CHAIN RULES]

$u = (2x-1)^2$ | $v = 3x+4$

$\dfrac{du}{dx} = 2(2x-1)(2) = 4(2x-1)$ | $\dfrac{dv}{dx} = 3$

QUOTIENT RULE

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(3x+4)(4(2x-1)) - (2x-1)^2(3)}{(3x+4)^2}$$

$$\frac{dy}{dx}_{(x=0)} = \frac{(3(0)+4)(4(2(0)-1)) - (2(0)-1)^2(3)}{(3(0)+4)^2} = \frac{-19}{16}$$

Q12 $y = (2x^2-3)^7$ find $\dfrac{dy}{dx}$ when $x=-1$ [CHAIN RULE]

$$\frac{dy}{dx} = 7(2x^2-3)^6(4x)$$

$$= 28x(2x^2-3)^6$$

$$\frac{dy}{dx}_{(x=-1)} = 28(-1)(2(-1)^2-3)^6 = (-28)(-1)^6 = -28$$

Q13 $f(x) = x\sqrt{x+1}$ $f'(x) = ?$ [PRODUCT AND CHAIN RULES]

$u = x$ | $v = (x+1)^{\frac{1}{2}}$

$\frac{du}{dx} = 1$ | $\frac{dv}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}}(1) = \frac{1}{2}(x+1)^{-\frac{1}{2}}$

Product Rule

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = (x)\left(\frac{1}{2}(x+1)^{-\frac{1}{2}}\right) + (x+1)^{\frac{1}{2}}(1)$$

$$= \frac{x}{2\sqrt{x+1}} + \frac{\sqrt{x+1}}{1} = \frac{x + 2(x+1)}{2\sqrt{x+1}} = \frac{3x+2}{2\sqrt{x+1}}$$

Q14

$4x^2 + 2xy = 5$ express y in terms of x and find $\frac{dy}{dx}$?

$2xy = 5 - 4x^2$

$y = \frac{5-4x^2}{2x}$

QUOTIENT RULE

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$u = 5 - 4x^2$ $v = 2x$

$\frac{du}{dx} = -8x$ $\frac{dv}{dx} = 2$

$$\frac{dy}{dx} = \frac{2x(-8x) - (5-4x^2)(2)}{(2x)^2} = \frac{-16x^2 - 10 + 8x^2}{4x^2}$$

$$= \frac{-8x^2 - 10}{4x^2} = \frac{-4x^2 - 5}{2x^2}$$

Q15 $y = \dfrac{x}{\sqrt{x}+1}$ find $\dfrac{dy}{dx}$ when $x=1$

$u = x$ $\quad v = \sqrt{x}+1 = x^{\frac{1}{2}}+1$

$\dfrac{du}{dx} = 1$ $\quad \dfrac{dv}{dx} = \frac{1}{2}x^{-\frac{1}{2}}+1 = \dfrac{1}{2\sqrt{x}}+1$

QUOTIENT RULE

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(\sqrt{x}+1)(1) - \left(\frac{1}{2\sqrt{x}}+1\right)}{(\sqrt{x}+1)^2}$$

$$\frac{dy}{dx}(x=1) = \frac{(\sqrt{1}+1)(1) - \left(\frac{1}{2\sqrt{1}}+1\right)}{(\sqrt{1}+1)^2} = \frac{(2)(1) - \left(\frac{1}{2}+1\right)}{(2)^2} = 2$$

Q16 $y = \dfrac{3x+1}{1-2x}$ express $\dfrac{dy}{dx}$ in the form $\dfrac{k}{(1-2x)^2}$ where k is constant.

$u = 3x+1$ $\quad v = 1-2x$

$\dfrac{du}{dx} = 3$ $\quad \dfrac{dv}{dx} = -2$

QUOTIENT RULE

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(1-2x)(3) - (3x+1)(-2)}{(1-2x)^2}$$

$$= \frac{3 - 6x + 6x + 2}{(1-2x)^2} = \frac{5}{(1-2x)^2}$$

$\Rightarrow k = 5$

Q17 $f(x) = \sqrt{3x^2 - 2}$ find $f'(x)$ when $x = 1$

CHAIN RULE

$$f(x) = (3x^2 - 2)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}(3x^2 - 2)^{-\frac{1}{2}}(6x) = 3x(3x^2 - 2)^{-\frac{1}{2}}$$

$$= \frac{3x}{\sqrt{3x^2 - 2}}$$

$$f'(1) = \frac{3(1)}{\sqrt{3(1)^2 - 2}} = 3$$

Q18 Show if $y = (x-1)^{3/2} - 3(x-1)^{\frac{1}{2}}$ then $\frac{dy}{dx} = \frac{3(x-2)}{2\sqrt{x-1}}$

CHAIN RULE

$$\frac{dy}{dx} = \frac{3}{2}(x-1)^{\frac{1}{2}}(1) - \frac{3}{2}(x-1)^{-\frac{1}{2}}(1)$$

$$= \frac{3}{2}\left(\frac{\sqrt{x-1}}{1} - \frac{1}{\sqrt{x-1}}\right)$$

$$= \frac{3}{2}\left(\frac{x-1-1}{\sqrt{x-1}}\right) = \frac{3(x-2)}{2\sqrt{x-1}}$$ QED

Q19 $y = ax^3 + 2bx^2 + 3cx$ $\quad \frac{dy}{dx} = 6x^2 + 6x - 6$

Find a, b and c ?

$$\frac{dy}{dx} = 3ax^2 + 4bx + 3c = 6x^2 + 6x - 6$$

$\Rightarrow 3a = 6$ $\quad a = 2$

$\Rightarrow 4b = 6$ $\quad b = \frac{3}{2}$

$\Rightarrow 3c = -6$ $\quad c = -2$

Q20 $y = \frac{x^2 - 5x + 4}{x}$ for $x > 0$, show $\frac{dy}{dx} = 1 - \frac{4}{x^2}$

$u = x^2 - 5x + 4$ $\quad v = x$

$\frac{du}{dx} = 2x - 5$ $\quad \frac{dv}{dx} = 1$

QUOTIENT RULE

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{x(2x-5) - (x^2-5x+4)(1)}{(x)^2}$$

$$= \frac{2x^2 - 5x - x^2 + 5x - 4}{x^2} = \frac{x^2 - 4}{x^2} = 1 - \frac{4}{x^2}$$

QED

Q21 $y = (3x + \sqrt{x})^2$ find $\frac{dy}{dx}$ and $\frac{dy}{dx}$ when $x = 1$

Chain Rule $\frac{dy}{dx} = 2(3x + \sqrt{x})\left(3 + \frac{1}{2}x^{-\frac{1}{2}}\right) = 2(3x + \sqrt{x})\left(3 + \frac{1}{2\sqrt{x}}\right)$

$\frac{dy}{dx}_{(x=1)} = 2(3(1) + \sqrt{1})\left(3 + \frac{1}{2\sqrt{1}}\right) = 2(4)\left(3\frac{1}{2}\right)$

$= 28$