11. Find S_n, the sum to n terms, of $1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots\ldots \left(\frac{1}{2}\right)^{n-1}$ and hence find S_∞, the sum to infinity of the series.
Find the least value of n such that $S_\infty - S_n < 0.001$.

$S_n = \frac{a(1-r^n)}{1-r}$

$a = 1 \quad r = \frac{1}{2} \quad n = n$

$S_n = \frac{1(1-(\frac{1}{2})^n)}{1-\frac{1}{2}} = \frac{1(1-(\frac{1}{2})^n)}{\frac{1}{2}} = 2(1-(\frac{1}{2})^n)$

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{1}{(1-\frac{1}{2})} = \frac{1}{(\frac{1}{2})} = 2$

$S_\infty - S_n < 0.001$

$\Rightarrow 2 - 2(1-(\frac{1}{2})^n) < 0.001$

Subtract 2

$-2(1-(\frac{1}{2})^n) < -1.999$

divide by -2 & change inequality

$1-(\frac{1}{2})^n > 0.9995$

Subtract 1

$-\frac{1}{2}^n > -0.0005$

change signs & inequality

$(\frac{1}{2})^n < 0.0005$

$\log_{\frac{1}{2}} 0.0005 = 10.965$

$\Rightarrow (\frac{1}{2})^{11} < 0.0005 \Rightarrow n = 11$

Section 4.6 Number Patterns

	Pattern	To find a
1st difference constant	$T_n = an + b$	a = 1st difference
2nd difference constant	$T_n = an^2 + bn + c$	$2a$ = 2nd difference
3rd difference constant	$T_n = an^3 + bn^2 + cn + d$	$6a$ = 3rd difference

Example 1

Express the nth term of the number pattern $-1, 13, 51, 125, 247, \ldots$ as a cubic polynomial.

	T_1	T_2	T_3		
	-1	13	51	125	247
1st D		14	38	74	122
2nd D			24	36	48
3rd D				12	12

a=?

Cubic

⇒ 6a= 3rd difference

$6a = 12 \Rightarrow a = 2$

Shape : $an^3 + bn^2 + cn + d$

$\Rightarrow 2n^3 + bn^2 + cn + d$

get 3 equations to be able to solve for 3 unknowns b, c and d.

$T_1 = 2(1)^3 + b(1)^2 + c(1) + d = -1$

$2 + b + c + d = -1$

$b + c + d = -3$ ①

$T_2 = 2(2)^3 + b(2)^2 + c(2) + d = 13$

$16 + 4b + 2c + d = 13$

$4b + 2c + d = -3$ ②

$T_3 = 2(3)^3 + b(3)^2 + c(3) + d = 51$

$54 + 9b + 3c + d = 51$

$9b + 3c + d = -3$ ③

Solve

$b + c + d = -3$ ①

$4b + 2c + d = -3$ ②

$9b + 3c + d = -3$ ③

②-① $3b + c = 0$ ④ $\Rightarrow c = -3b$

③-① $8b + 2c = 0$ ⑤

④→⑤ $8b + 2(-3b) = 0$

$8b - 6b = 0$

$2b = 0 \Rightarrow b = 0$

→④ $c = 3(0) \Rightarrow c = 0$

→① $0 + 0 + d = -3 \Rightarrow d = -3$

General expression $2n^3 + bn^2 + cn + d$

Cubic polynomial $= 2n^3 - 3$

6. Each of the following number patterns can be written in the form $an^3 + bn^2 + cn + d$. Find the values of a, b, c, and d in each case:

(i) 6, 27, 74, 159, 294
(ii) 3, −1, −1, 9, 35
(iii) −1, 2, 17, 50, 107

	T_1	T_2	T_3		
	6	27	74	159	294
Differences D_1		21	47	85	135
D_2			26	38	50
D_3				12	12

for cubic 6a= 3rd difference

$a = ?$ $6a = 12 \Rightarrow a = 2$

cubic shape $an^3 + bn^2 + cn + d = 2n^3 + bn^2 + cn + d$

$T_1 =$ $2(1)^3 + b(1)^2 + c(1) + d = 6$

$2 + b + c + d = 6$

$b + c + d = 4$ ①

$T_2 =$ $2(2)^3 + b(2)^2 + c(2) + d = 27$

$16 + 4b + 2c + d = 27$

$4b + 2c + d = 11$ ②

$T_3 =$ $2(3)^3 + b(3)^2 + c(3) + d = 74$

$54 + 9b + 3c + d = 74$

$9b + 3c + d = 20$ ③

Solve

$b + c + d = 4$ ①

$4b + 2c + d = 11$ ②

$9b + 3c + d = 20$ ③

② − ①: $3b + c = 7 \quad = \quad 3b + c = 7$ ④

③ − ①: $8b + 2c = 16 \Rightarrow 4b + c = 8$ ⑤

⑤ − ④: $\Rightarrow b = 1$

→ ④: $\Rightarrow 3(1) + c = 7 \Rightarrow c = 4$

→ ①: $\Rightarrow 1 + 4 + d = 4$

$5 + d = 4 \Rightarrow d = -1$

cubic shape $2n^3 + bn^2 + cn + d$

$= 2n^3 + n^2 + 4n - 1$ ✓

6. Each of the following number patterns can be written in the form $an^3 + bn^2 + cn + d$. Find the values of $a, b, c,$ and d in each case:

(i) 6, 27, 74, 159, 294
(ii) 3, −1, −1, 9, 35
(iii) −1, 2, 17, 50, 107

	T_1	T_2	T_3		
	3	−1	−1	9	35
Differences D_1		−4	0	10	26
D_2			4	10	16
D_3				6	6

for cubic 6a= 3rd difference $\quad a = ? \quad 6a = 6 \Rightarrow a = 1$

cubic shape $\quad an^3 + bn^2 + cn + d = n^3 + bn^2 + cn + d$

$T_1 =$ $(1)^3 + b(1)^2 + c(1) + d = 3$
$1 + b + c + d = 3$
$b + c + d = 2 \quad (1)$

$T_2 =$ $(2)^3 + b(2)^2 + c(2) + d = -1$
$8 + 4b + 2c + d = -1$
$4b + 2c + d = -9 \quad (2)$

$T_3 =$ $(3)^3 + b(3)^2 + c(3) + d = -1$
$27 + 9b + 3c + d = -1$
$9b + 3c + d = -28 \quad (3)$

Solve

$b + c + d = 2 \quad (1)$
$4b + 2c + d = -9 \quad (2)$
$9b + 3c + d = -28 \quad (3)$

(2)−(1) $\quad 3b + c = -11 \quad (4)$
(3)−(1) $\quad 8b + 2c = -30 \Rightarrow 4b + c = -15 \quad (5)$

(5)−(4) $\quad \Rightarrow b = -4$

→(4) $\quad \Rightarrow 3(-4) + c = -11$
$-12 + c = -11 \Rightarrow c = 1$

→(1) $\quad \Rightarrow -4 + 1 + d = 2$
$-3 + d = 2 \Rightarrow d = 5$

cubic shape $\quad n^3 + bn^2 + cn + d$

$= n^3 - 4n^2 + n + 5$ ✓

6. Each of the following number patterns can be written in the form $an^3 + bn^2 + cn + d$. Find the values of a, b, c, and d in each case:

(i) 6, 27, 74, 159, 294
(ii) 3, −1, −1, 9, 35
(iii) −1, 2, 17, 50, 107

	T_1	T_2	T_3		
	−1	2	17	50	107
Differences D_1		3	15	33	57
D_2			12	18	24
D_3				6	6

for cubic 6a= 3rd difference

$a = ?$ $\quad 6a = 6 \Rightarrow a = 1$

cubic shape

$an^3 + bn^2 + cn + d = n^3 + bn^2 + cn + d$

$T_1 =$

$(1)^3 + b(1)^2 + c(1) + d = -1$

$1 + b + c + d = -1$

$b + c + d = -2$ ①

$T_2 =$

$(2)^3 + b(2)^2 + c(2) + d = 2$

$8 + 4b + 2c + d = 2$

$4b + 2c + d = -6$ ②

$T_3 =$

$(3)^3 + b(3)^2 + c(3) + d = 17$

$27 + 9b + 3c + d = 17$

$9b + 3c + d = -10$ ③

Solve

$b + c + d = -2$ ①

$4b + 2c + d = -6$ ②

$9b + 3c + d = -10$ ③

② − ① $\quad 3b + c = -4$ ④

③ − ① $\quad 8b + 2c = -8 \Rightarrow 4b + c = -4$ ⑤

⑤ − ④ $\quad \Rightarrow b = 0$

→ ④ $\quad \Rightarrow 3(0) + c = -4 \Rightarrow c = -4$

→ ① $\quad \Rightarrow 0 - 4 + d = -2$

$d = 2$

cubic shape

$n^3 + bn^2 + cn + d$

$= n^3 - 4n + 2$ ✓