Section 4.4 Geometric sequences

A **geometric sequence** is formed when each term of the sequence is obtained by multiplying the previous term by a fixed amount.

For example, $2, \overset{\times 3}{\rightarrow} 6, \overset{\times 3}{\rightarrow} 18, \overset{\times 3}{\rightarrow} 54, \ldots\ldots$ each term increasing by a factor of 3.

$4, \overset{\times \frac{1}{2}}{\rightarrow} 2, \overset{\times \frac{1}{2}}{\rightarrow} 1, \overset{\times \frac{1}{2}}{\rightarrow} \frac{1}{2}, \ldots\ldots$ each term decreasing by a factor of $\frac{1}{2}$.

For any geometric sequence, the first term is denoted by a and the ratio between consecutive terms is r (called the common ratio); then every geometric sequence can be represented by

$$\underset{a,}{T_1,} \quad \overset{+r}{\underset{ar,}{T_2,}} \quad \overset{+r}{\underset{ar^2,}{T_3,}} \quad \overset{+r}{\underset{ar^3,}{T_4,}} \quad \overset{+r}{\underset{ar^4,}{T_5,}} \quad \ldots\ldots \quad \underset{ar^{n-1}}{T_n}$$

In every geometric sequence:

$$T_1 = a$$

$$\rightarrow \frac{T_2}{T_1} = r$$

$$T_n = ar^{n-1}$$

$$\frac{T_{n+1}}{T_n} = r$$

Geometric Sequence

Rule?

$\overset{\times 2}{\rightarrow} 2, \overset{\times 2}{\rightarrow} 4, \overset{\times 2}{\rightarrow} 8, 16, \ldots\ldots$

a

$T_6 = ?$

$T_1 = a$

$T_2 = 2^1 a$

$T_3 = 4a = a2^2$

$T_4 = a2^3$

$\vdots$

$T_{10} = a2^9$

$T_{99} = a2^{98}$

General

$$\boxed{T_n = ar^{n-1}}$$

Example 1

Find T_n and T_{10} of the geometric sequence $1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots$

$T_n = ar^{n-1}$

$r = \frac{T_{n+1}}{T_n}$

$a = 1$, $r = \frac{(\frac{1}{4})}{1} = \frac{1}{4}$

$T_n = 1(\frac{1}{4})^{n-1}$ ✓

$T_n = \frac{1}{4^{n-1}}$

$T_{10} = \frac{1}{4^9} = \frac{1}{262144}$ ✓

Example 2

In a geometric sequence, $T_3 = 32$ and $T_6 = 4$.

Find a and r and hence write down the first six terms of the sequence.

$T_n = ar^{n-1}$

$T_3 = 32$

$32 = ar^2$ (1) $a = 32/r^2$

$T_6 = 4$

$4 = ar^5$ (2)

(1)→(2) $4 = \left(\frac{32}{r^2}\right)r^5 = 32r^3$

$r^3 = \frac{4}{32} = \frac{1}{8}$

$r = \sqrt[3]{\frac{1}{8}} = \frac{1}{2}$ $r = \frac{1}{2}$

$a = 32/(\frac{1}{2})^2 = 128 = a$

1st 6 Terms: 128, 64, 32, 16, 8, 4

3. Given $T_2 = 12$ and $T_5 = 324$, find a and r and hence write down the first five terms of the sequence.

$T_n = ar^{n-1}$

Solve

$T_2 = 12 \Rightarrow 12 = ar^1 \Rightarrow a = 12/r$ (1)

$T_5 = 324 \Rightarrow 324 = ar^4$ (2)

(1) → (2)

$324 = \left(\frac{12}{r}\right) r^4$

$r^3 = \frac{324}{12} = 27 \Rightarrow r = \sqrt[3]{27} \Rightarrow r = 3$

(1) $a = 12/3 = 4$, $a = 4$

First 5 terms: $4, 12, 36, 108, 324$

Exercise 4.4

1. Determine which of the following sequences are geometric.

Find the common ratios of these sequences and write down the next two terms of each sequence.

(i) 3, 9, 27, 81, ...

(ii) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

$r = \frac{T_2}{T_1}$

To get next term x3

(i) $r = \frac{9}{3} = 3$

$3, 9, 27, 81, \underline{243}, \underline{729}$

(ii) $r = \frac{(1/3)}{1} = \frac{1}{3}$

$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \underline{\frac{1}{81}}, \underline{\frac{1}{243}}$

2. Each of the following sequences is geometric.
Find a and r and hence find the indicated term.

(i) 5, 10, … (T_{11}) (ii) 10, 25, … (T_7)

$r = \frac{T_2}{T_1}$

$T_n = ar^{n-1}$

(i) $a = 5$, $r = \frac{10}{5} = 2$

$T_{11} = 5(2)^{10} = 5,120$

(ii) $a = 10$, $r = \frac{25}{10} = 2.5$

$T_7 = 10(2.5)^6 = 2441.40625$

5. Write down the first five terms of the geometric sequence that has a second term 4 and a fifth term $-\frac{1}{16}$.

$T_n = ar^{n-1}$

$T_2 = 4 \Rightarrow 4 = ar^1 \Rightarrow a = 4/r$ (1)

$T_5 = -\frac{1}{16} \Rightarrow -\frac{1}{16} = ar^4$ (2)

SOLVE

(1) → (2) $-\frac{1}{16} = \left(\frac{4}{r}\right)r^{4/3} \Rightarrow -\frac{1}{16} = 4r^3$

÷4 $-\frac{1}{64} = r^3 \Rightarrow r = \sqrt[3]{\left(-\frac{1}{64}\right)} \Rightarrow r = -\frac{1}{4}$

$a = ?$ $T_2 = ar^1$ $T_2 = 4 \Rightarrow 4 = a\left(-\frac{1}{4}\right) \Rightarrow -16 = a$

First 5 terms $-16, 4, -1, \frac{1}{4}, -\frac{1}{16}$

7. The three numbers $n-2, n$ and $n+3$ are three consecutive terms of a geometric sequence. Find the value of n and hence write down the first four terms of the sequence.

$r = \frac{T_{n+1}}{T_n}$

Solve

$r = \frac{n}{n-2} = \frac{n+3}{n}$

$\Rightarrow \quad n^2 = (n+3)(n-2)$

$n^2 = n^2 - 2n + 3n - 6$

$0 = n - 6$

$n = 6$

$n = 6$

$a = 4$

$\Rightarrow \quad n-2 = 6-2 = 4 \quad \therefore a = 4$

$n = 6$

$n+3 = 6+3 = 9$

$r = \frac{T_{n+1}}{T_n}$

$r = \frac{6}{4} \quad \Rightarrow r = 3/2$

First 4 terms: $4, 6, 9, 13.5$