Exponential sequences

Exponential functions of the form $y = Aa^x$, where A is the initial value and a the multiplier or common ratio, produce geometric sequences.

Consider a ball dropping from a height of 10 m.

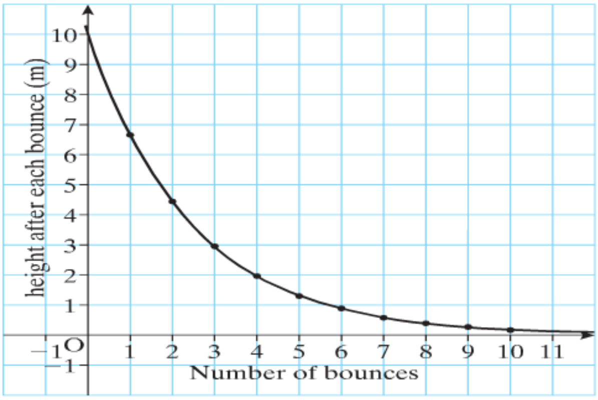
If the ball bounces back to $\frac{2}{3}$ of its original height on each bounce, the height of the ball is given by the following pattern:

After 1 bounce: $10 \times \frac{2}{3} = 10\left(\frac{2}{3}\right)^1$

After 2 bounces: $10 \times \frac{2}{3} \times \frac{2}{3} = 10\left(\frac{2}{3}\right)^2$

After 3 bounces: $10 \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = 10\left(\frac{2}{3}\right)^3$

After n bounces: $10 \times \left(\frac{2}{3}\right)^n$

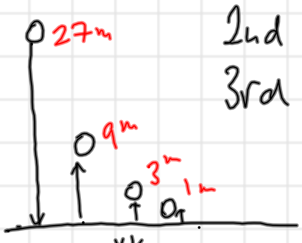


Example 6

A ball is dropped from a height of 27 m and loses $\frac{2}{3}$ of its height on each bounce.

(i) Find the height of the ball on each of its first four bounces. ✓
(ii) Hence write down the height of the ball after the 10th bounce. ✓
(iii) After which bounce will the ball be at most 2.5 m above the ground?

Picture



2nd → 3 m
3rd → 1 m

$\times \frac{1}{3}$

Sequence: $27, 9, 3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27} \ldots$

$a = 27$, $r = \frac{9}{27} = \frac{1}{3}$

$T_n = ar^{n-1}$

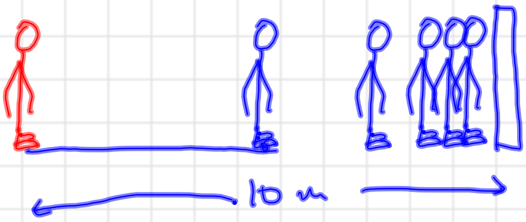
$T_{10} = 27\left(\frac{1}{3}\right)^9 = \frac{1}{729}$

For a geometric series with $|r| < 1$,
$$\lim_{n \to \infty} S_n = \frac{a}{1-r}.$$

$|r| < 1$

The idea of a sum of infinite terms having a limit.

If I walk towards a wall that is 10 m away and every second I cover half the distance between me and the wall. I will never reach the wall. The sum of all the distances I cover will add up to slightly less than 10 m!



For a geometric series with $|r| < 1$,
$$\lim_{n \to \infty} S_n = \frac{a}{1-r}.$$

Example 3

Find the sum to infinity of the geometric series $16 + 12 + 9 + \ldots..$

$a = 16$ $\qquad r = \frac{12}{16} = \frac{3}{4}$

$$S_\infty = \frac{a}{1-r} = \frac{16}{1-\frac{3}{4}} = \frac{16}{\frac{1}{4}} = 64$$

19. The value of a sum of money on deposit at 3% per annum compound interest is given by $A = €4000\,(1.03)^t$ where t is the number of years of the investment. Find

(i) the amount of money on deposit
(ii) the value of the investment at the end of each of the first four years
(iii) the value of the investment at the end of the 10th year
(iv) the number of years, correct to the nearest year, needed for the investment to double in value.

(i) on deposit $\Rightarrow t = 0$ years

$\Rightarrow A_0 = 4000\,(1.03)^0 = €\,4000$

(ii) 1 year $\Rightarrow A_1 = 4000(1.03)^1 = €\,4120$
2 year $\Rightarrow A_2 = 4000\,(1.03)^2 = €\,4243.60$
3 year $\Rightarrow A_3 = 4000\,(1.03)^3 = €\,4370.91$
4 years $\Rightarrow A_4 = 4000\,(1.03)^4 = €\,4502.04$

(iii) $A_{10} = 4000\,(1.03)^{10} = €\,5375.67$

$A_0 = €4000$
Double $A_0 = €8000$
$t = ?$

$A = 4000\,(1.03)^t$
$\Rightarrow 8000 = 4000\,(1.03)^t$
$\Rightarrow 2 = 1.03^t$
$t = \log_{1.03} 2 = 23.45 \simeq 23$ years (n.w.n)

Recurring decimals

Recurring decimals can be expressed as a sum to infinity of a geometric sequence, where the common ratio $r < 1$.

For example, $0.\dot{3} = 0.3333\ldots\ldots = \frac{3}{10} \xrightarrow{\times\frac{1}{10}} + \frac{3}{10^2} \xrightarrow{\times\frac{1}{10}} + \frac{3}{10^3} \xrightarrow{\times\frac{1}{10}} + \frac{3}{10^4} + \ldots\ldots$

where $a = 0.3$ and $r = \frac{1}{10}$.

Similarly,

$$0.2\dot{3}\dot{5} = 0.2353535\ldots\ldots = 0.2 + [0.035 + 0.00035 + \ldots\ldots]$$
$$= 0.2 + \frac{35}{1000} + \frac{35}{100000} + \ldots\ldots$$
$$= 0.2 + \text{an infinite geometric series}$$

where $a = \frac{35}{1000}$ and $r = \frac{1}{100}$.

10. Write each of the following recurring decimals as an infinite geometric series. Hence express each as a decimal in the form $\frac{a}{b}$, $a, b \in N$.

(i) $0.\dot{7}$ (ii) $0.\dot{3}\dot{5}$ (iii) $0.2\dot{3}$ (iv) $0.\dot{3}7\dot{0}$ (v) $0.1\dot{6}\dot{2}$ (vi) $0.3\dot{2}\dot{1}$

Ex.4.5

(i) $0.\dot{7} = 0.7777\ldots = \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \ldots$

$S_\infty = \frac{a}{1-r}$

$a = \frac{7}{10}$ $r = \frac{1}{10}$

$S_\infty = \frac{7/10}{1 - 1/10} = \frac{7/10}{9/10} = \frac{7}{9}$

(ii) $0.\dot{3}\dot{5} = 0.353535\ldots = \frac{35}{100} + \frac{35}{10000} + \frac{35}{1000000} + \ldots$

$a = 35/100$ $r = 1/100$

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{(35/100)}{(1 - 1/100)} = \frac{(35/100)}{(99/100)} = \frac{35}{99}$