chapter

10 Patterns and Sequences

Key words

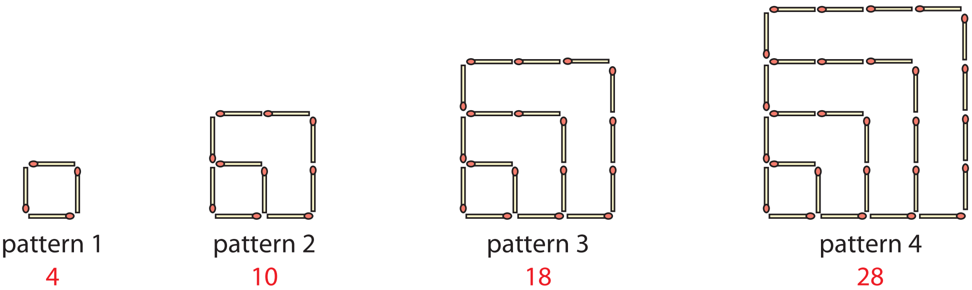
pattern **sequence** **term** **method of difference** ***n*th term**
coefficient **arithmetic sequence** **common difference** **arithmetic series**
S_n, the sum to *n* terms **quadratic sequence**

Section 10.1 Patterns in number

The ability to see patterns or sequences is very important in mathematics.
We meet number patterns such as 1, 3, 5, 7, … or 5, 10, 15, 20, … on a regular basis.

We see **patterns** in designs such as tiling and mosaics.

Here is a growth pattern of squares made from matchsticks.



The numbers in red below each pattern represent the numbers of matches used in each shape.
The numbers 4, 10, 18, 28, … generated by these patterns form a sequence that is a little more complex than the sequences 1, 3, 5, 7, … or 5, 10, 15, 20, … .

Number sequences

A number sequence is an ordered set of numbers with a rule to find every number in the sequence. The rule which takes you from one number to the next could be a simple addition or multiplication, but generally it is more tricky than that. In more difficult sequences you need to examine them carefully to identify the pattern.

Each number in a sequence is called a **term**.
The first term is written as T_1 ; the 4th term is T_4.

Look at these sequences and their rules.

4, 8, 16, 32, … doubling the preceding term each time … 64, 128, …
4, 7, 10, 13, … adding 3 to the preceding term each time … 16, 19, …
36, 32, 28, 24, … subtracting 4 from the preceding term each time … 20, 16, …

These sequences are all quite straightforward once you have found the link from one term to the next.

Method of differences

For some sequences that are not immediately obvious, we need to look at the difference between consecutive terms to determine the pattern.

Consider the sequence: 3, 6, 11, 18, 27
Difference between terms: 3, 5, 7, 9

Here the differences form a sequence of their own.
The pattern is much easier to detect in this sequence.
The next difference is 11.
We now find the next term of the first sequence by adding 11 to 27.
The next term is $27 + 11$, i.e., 38.

Exercise 10.1

1. Look at the following number sequences.
Write down the next three terms in each and explain how the sequence is found.

(i) 2, 4, 6, 8, …
(ii) 1, 3, 5, 7, …
(iii) 1, 4, 7, 10, …
(iv) 1, 2, 4, 8, …
(v) 3, 9, 27, …
(vi) 16, 8, 4, …
(vii) 20, 18, 16, …
(viii) 2, 6, 18, …

2. By considering the differences in the following sequences, write down the next two terms in each case:

(i) 2, 4, 7, 11, …
(ii) 1, 2, 5, 10, …
(iii) 2, 6, 12, 20, …
(iv) 2, 3, 6, 11, 18, …
(v) 1, 4, 10, 19, …
(vi) 2, 7, 14, 23, …

3. Look carefully at each number sequence below.
Find the next two terms in the sequence and try to explain the pattern.

(i) 1, 1, 2, 3, 5, 8, 13, …
(ii) 3, 4, 7, 11, 18, 29, …
(iii) 1, 8, 27, 64, …
(iv) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

4. Examine these number patterns:
Write down the next line in each pattern without using a calculator.
Now use a calculator to check that you are correct.

(i)
$6 \times 9 = 54$
$66 \times 9 = 594$
$666 \times 9 = 5994$
$= = = = = =$

(ii)
$9 \times 1 = 9$
$9 \times 12 = 108$
$9 \times 123 = 1107$
$9 \times 1234 = 11106$
$= = = = = = = =$

(iii)
$7 \times 7 = 49$
$67 \times 67 = 4489$
$667 \times 667 = 444889$
$6667 \times 6667 = 44448889$
$= = = = = = = = = =$

Section 10.2 The *n*th term of a sequence

When using a number sequence, we sometimes need to know, for example, the 50th or 100th term without having to write out **all** 50 or 100 terms. To do this we need to find the rule which generates the sequence.

This rule is generally called the ***n*th** term or T_n.

If $T_n = 2n + 3$, then $T_1 = 2(1) + 3 = 5$
$T_2 = 2(2) + 3 = 7$
$T_3 = 2(3) + 3 = 9$
....

From this we can see that $T_n = 2n + 3$ generates the sequence 5, 7, 9, …
The rule, $T_n = 2n + 3$, allows us to find any term of the sequence.

Example 1

The *n*th term of a sequence is $4n - 3$.
Write down the first five terms of the sequence.

$T_n = 4n - 3$
$T_1 = 4(1) - 3 = 1$
$T_2 = 4(2) - 3 = 5$
$T_3 = 4(3) - 3 = 9$
$T_4 = 4(4) - 3 = 13$
$T_5 = 4(5) - 3 = 17$

The first 5 terms are: 1, 5, 9, 13, 17.

Finding the *n*th term of a sequence

Consider the sequence 2, 5, 8, 11, …
Here we have the **same difference** between any term and the next.
This difference is **3**.
The nth term of the sequence will be **3***n* $\pm$ a number.
3*n* will generate the sequence 3, 6, 9, 12.

Compare the new sequence with the original one and see what number you need to add to or subtract from each term to get the original pattern.

$$\begin{array}{c l} -1 & 2,\ 5,\ 8,\ 11 \\ & 3,\ 6,\ 9,\ 12 \end{array}$$

Here we subtract one from each term to get the original term.

So the nth term is $3n - 1$;

$\therefore\ T_n = 3n - 1$

In the sequence 6, 10, 14, 18, … the difference is **4**.
The nth term will be $T_n = \mathbf{4}n \pm$ a number.
$4n$ generates the sequence 4, 8, 12, …
Comparing the original sequence with the new sequence we have:

$$\begin{array}{c l} +2 & 6,\ 10,\ 14,\ 18,\ \ldots \\ & 4,\ \ 8,\ 12,\ 16,\ \ldots \end{array}$$

Here we add 2 to each term to get the corresponding term of the original sequence.

The nth term is $T_n = 4n + 2$

Example 2

Find the nth term of the sequence 3, 7, 11, 15, … .

In the sequence 3, 7, 11, 15, …, the difference between the terms is 4.
So the first part of the nth term is $4n$.
This would generate the sequence 4, 8, 12, 16, …

Comparing the sequences we have: $\begin{array}{l c} 3,\ 7,\ 11,\ 15 & -1 \\ 4,\ 8,\ 12,\ 16 & \end{array}$

Here we subtract 1 from each term to get the original sequence.

$\therefore\ T_n = 4n - 1$

Example 3

Find the nth term and the 20th term of the sequence 4, 7, 10, 13, 16, …

In the sequence 4, 7, 10, 13, …, the difference between the terms is **3**.

$T_n = 3n \pm$ a number

$3n$ generates the sequence 3, 6, 9, 12, …

Comparing: $\begin{array}{c l} +1 & 4,\ 7,\ 10,\ 13,\ \ldots \\ & 3,\ 6,\ \ 9,\ 12,\ \ldots \end{array}$

Here we add 1 to each term of the second sequence to get the first sequence.

$\therefore\ T_n = 3n + 1$

$T_{20} = 3(20) + 1$ … substitute 20 for n
$T_{20} = 61$

Exercise 10.2

1. Use each of the following rules to write down the first three terms of the sequence:

(i) $n + 4$ for $n = 1, 2, 3$ (ii) $2n + 1$ for $n = 1, 2, 3$

(iii) $4n + 3$ for $n = 1, 2, 3$ (iv) $3n - 2$ for $n = 1, 2, 3$.

2. In each of the following sequences the nth term, T_n, is given.
Write down the first four terms of each sequence:

(i) $T_n = 3n$ (ii) $T_n = 2n + 3$ (iii) $T_n = 3n - 2$

3. If $T_n = 3n - 4$, find T_1, T_3 and T_{10}.

4. Write down the first three terms of each of these sequences where the nth term is given.

(i) $T_n = n^2$ (ii) $T_n = n^2 + 3$ (iii) $T_n = 2n^2 + 1$

5. If $T_n = 2n^2 - 1$, work out

(i) T_1 (ii) T_2 (iii) T_5 (iv) T_{10}.

6. If $T_n = 2n - 6$, show that $T_1 + T_5 = 0$.

7. Find the next two terms and the nth term of each of the following sequences:

(i) 3, 5, 7, 9, 11, … (ii) 4, 7, 10, 13, …

(iii) 2, 6, 10, 14, 18, … (iv) 5, 9, 13, 17, …

8. Find the nth term and the 20th term of each of these sequences:

(i) 2, 5, 8, 11, … (ii) 6, 8, 10, 12, …

(iii) 4, 9, 14, 19, … (iv) 2, 7, 12, 17, …

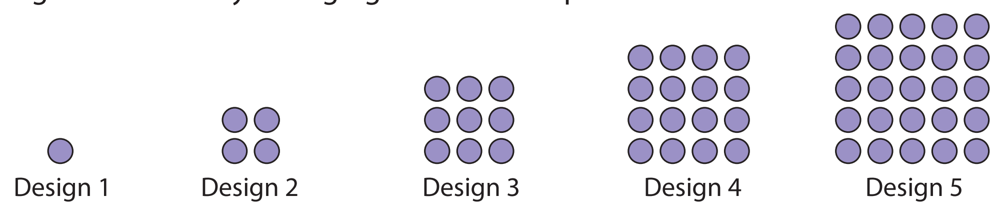
9. For the sequence 6, 11, 16, 21, …
find (i) T_n (ii) T_{20} (iii) T_{100}

10. The nth term of a sequence is given by $T_n = 3n - 4$.
Which term of the sequence is 23?

[Hint: Let $3n - 4 = 23$.]

Section 10.3 Sequences from shapes

These designs are made by arranging counters in squares.



The number of counters in each design is shown in this table.

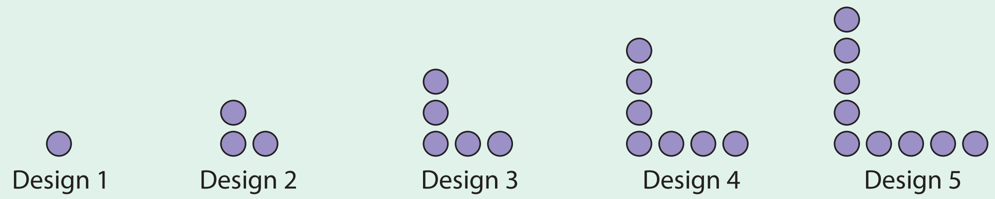
Design number	1	2	3	4	5
Number of counters	1	4	9	16	25

The numbers in the sequence 1, 4, 9, 16, 25 … are called **square numbers**.

From the sequence, it can be seen that the number of counters required for design 6 is 6^2, i.e. 36, and for design n, it is n^2.

Example 1

These designs are made by arranging counters in L-shapes.



(i) Copy and complete this table for these designs.

Design number	1	2	3	4	5
Number of counters	1				

(ii) How many counters are in the 6th design?
(iii) How many counters are needed to make the 15th design? Explain how you worked out your answer.
(iv) Which design uses 99 counters?
(v) Is it possible to make one of these designs with exactly 40 counters. Explain your answer.

(i) Here is the table:

Design number	1	2	3	4	5
Number of counters	1	3	5	7	9

(ii) There will be $9 + 2$, i.e. 11, counters in the 6th design.
(iii) To find the number of counters in the 15th design we need to find the nth term.

1, 3, 5, 7, 9, … . Here the difference is 2.

So the nth term will be $2n \pm$ a number.
$2n$ generates 2, 4, 6, 8, … so we need to take 1 from each term of this sequence to get the first sequence.

$$\therefore\ T_n = 2n - 1$$

The 15th design is represented by T_{15}.

$$T_n = 2n - 1 \Rightarrow T_{15} = 2(15) - 1 = 29$$

So 29 counters are needed for the 15th design.

(iv) To find which design uses 99 counters, we let $T_n = 99$.

$$T_n = 99 \Rightarrow 2n - 1 = 99$$
$$\Rightarrow \quad 2n = 100$$
$$\Rightarrow \quad n = 50$$

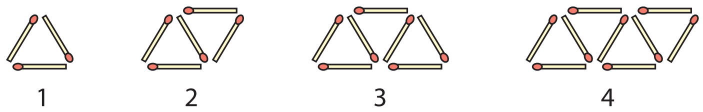
Design number 50 uses 99 counters.

(v) Let
$$T_n = 40$$
$$2n - 1 = 40$$
$$2n = 41$$
$$n = 20\tfrac{1}{2}$$

Since n is not a whole number, no design uses exactly 40 counters.

Exercise 10.3

1. A pattern of triangles is built up from matchsticks.



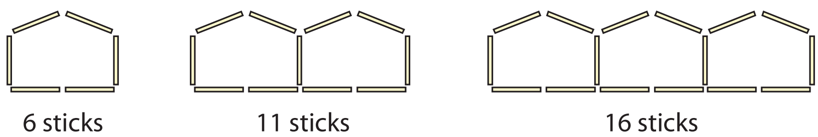
(i) Draw the 5th set of triangles in this pattern.

(ii) Write down the sequence of numbers generated by the matchsticks in the first six patterns.

(iii) Find an expression in n for the number of matches in the nth set of triangles.

(iv) How many matches are needed for the 50th set of triangles?

2. Here is a pattern sequence



(i) Draw the 4th pattern in this sequence.

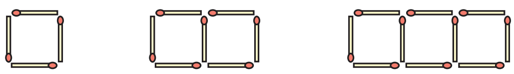
(ii) Write down the sequence of numbers generated by the sticks in the first six patterns.

(iii) Show that the number of sticks in the nth pattern is given by $T_n = 5n + 1$.

(iv) How many sticks are required for the 20th pattern?

(v) For which pattern are 51 sticks required?

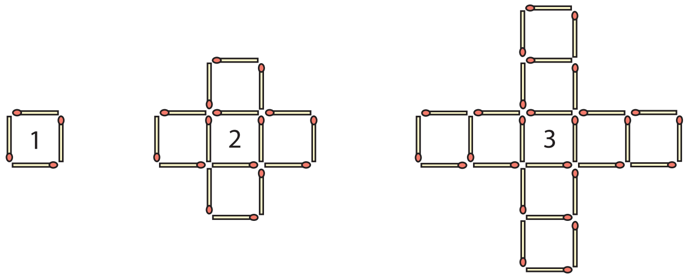
3. Complete the table of values for this sequence of matchstick patterns.



Number of squares	1	2	3	4	5
Number of matchsticks	4	7			

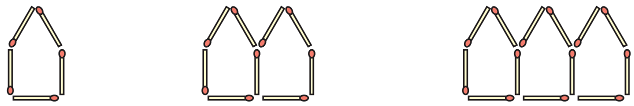
(i) How many matchsticks are required for the 6th pattern?
(ii) Find an expression in n for the nth pattern.
(iii) Use the expression found to find the number of matchsticks required for the 50th pattern.

4. A pattern of squares is built up from matchsticks as shown.



(i) Draw the 4th pattern.
(ii) Find an expression in n for the number of squares in the nth pattern.
(iii) How many squares are there in the 30th pattern?
(iv) Which pattern contains exactly 77 squares?

5. Look at these matchstick shapes.


5 matchsticks   9 matchsticks

(i) Copy and complete the table below:

Shape number	1	2	3	4	5
Number of matchsticks	5	9	…	…	…

(ii) How many matchsticks are there in Shape 7?
(iii) Find an expression for the number of matchsticks in Shape n.
(iv) Which shape contains exactly 101 matchsticks?

6. (i) Find the eighth term of the sequence whose nth term is $4n - 1$.
(ii) Find the nth term of the sequence whose first four terms are

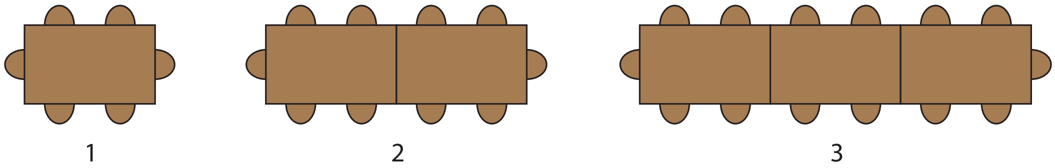
2 8 14 20

7. Each of these patterns uses black tiles.
(i) How many black tiles will be in pattern 5?
(ii) How many black tiles will be in pattern 10?


Pattern 1 Pattern 2 Pattern 3 Pattern 4

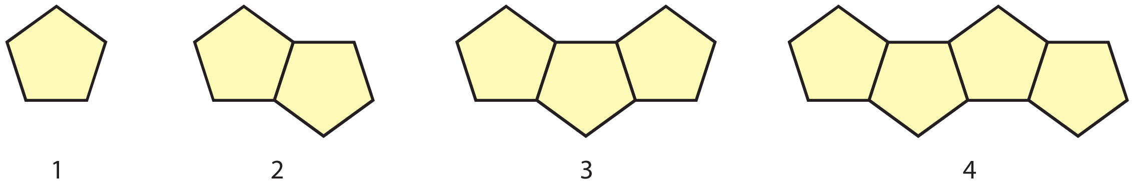
(iii) Find an expression for the number of black tiles in pattern n.
(iv) How many tiles will be in pattern 100?
(v) Which pattern will have exactly 101 tiles?

8. A conference centre had tables each of which could sit six people.
When put together, the tables could seat people as shown.



(i) How many people could be seated at 4 tables?
(ii) How many people could be seated at n tables put together in this way?
(iii) A conference had 90 people who wished to use the tables in this way.
How many tables would they need?

9. Regular pentagons of side length 1 cm are joined together to make a pattern as shown.



(i) Write down the perimeter of each of the first 4 shapes.
(Do not include internal lines.)
(ii) What is the perimeter of the 5th and 6th shapes?
(iii) Find an expression for the perimeter of the nth shape.
(iv) Find the length of the perimeter of the 50th shape.
(v) Which shape has a perimeter of length 92 cm?

Section 10.4 Arithmetic sequences

Consider these sequences: (i) 2, 4, 6, 8, 10, …
(ii) 3, 7, 11, 15, 19, …

In (i) each term is found by adding 2 to the preceding term.
In (ii) each term is found by adding 4 to the preceding term.
These are examples of **arithmetic sequences**.

Arithmetic sequence

A sequence, in which any term after the first can be obtained by adding a fixed number to the term before it, is called an arithmetic sequence.

The first term is denoted by $\boldsymbol{a}$.
The fixed number is called the **common difference** and is denoted by the letter $\boldsymbol{d}$.

Here are some examples of arithmetic sequences:

	First term (a)	Common difference (d)
(i) 5, 8,11, …	5	3
(ii) 6, 3, 0, …	6	-3
(iii) -4, 0, 4, …	-4	4

The common difference **d = any term − previous term**.

In the sequence 12, 7, 2, …

$d = 7 - 12 = -5$.

Since d is negative, the sequence is decreasing.

Finding the *n*th term of an arithmetic sequence

If a is the first term of an arithmetic sequence and d is the common difference, the sequence may be written as follows:

a,	$a + d$,	$a + 2d$,	$a + 3d$,	…	$a + (n-1)d$
T_1	T_2	T_3	T_4	…	T_n

The nth term of an arithmetic sequence is given by $T_n = a + (n-1)d$

Example 1

In the arithmetic sequence 3, 8, 13, …, find

(i) a (ii) d (iii) T_n (iv) T_{20}

(i) a, the first term = 3

(ii) d, the common difference = any term − previous term = 8 − 3 = 5

(iii) $T_n = a + (n-1)d$
$= 3 + (n-1)5$
$= 3 + 5n - 5$
$= 5n - 2$

(iv) $T_n = 5n - 2$
$T_{20} = 5(20) - 2$
$= 98$

Example 2

(i) Find the nth term of the arithmetic sequence 7, 10, 13, 16, …

(ii) Which term of the sequence is 97?

(iii) Show that 168 is not a term of the sequence.

(i) 7, 10, 13, 16, … $\quad a = 7$ and $d = 3$

$$\begin{aligned} T_n &= a + (n-1)d \\ &= 7 + (n-1)3 \\ &= 7 + 3n - 3 \\ T_n &= 3n + 4 \end{aligned}$$

(ii) Let $T_n = 97 \Rightarrow 3n + 4 = 97$

$$\begin{aligned} 3n &= 97 - 4 = 93 \\ n &= 31 \end{aligned}$$

The 31st term is 97.

(iii) Let $T_n = 168$

$$\begin{aligned} 3n + 4 = 168 \Rightarrow 3n &= 168 - 4 \\ 3n &= 164 \\ n &= \frac{164}{3} = 54\tfrac{2}{3} \end{aligned}$$

Since n is not a whole number, 168 is not a term of the sequence.

Exercise 10.4

1. Find a and d for each of the following arithmetic sequences:
(i) 2, 5, 8, … (ii) 7, 12, 17, … (iii) 0, 3, 6, 9, …
(iv) −2, 1, 4, … (v) 60, 55, 50, … (vi) 6, 1, −4, …

2. Write out the next three terms of the following arithmetic sequences:
(i) 2, 6, 10, … (ii) 8, 4, 0, … (iii) −6, −3, 0, …

3. In the arithmetic sequence 2, 6, 10, 14, … write down
(i) a, the first term (ii) d, the common difference
(iii) T_n and hence the value of T_{20}

4. Find T_n, the nth term of each of the following sequences:
(i) 1, 5, 9, 13, … (ii) 6, 8, 10, 12, … (iii) −5, 0, 5, 10, …

5. If T_n of a sequence is $3n - 4$, write down the value of
(i) T_1 (ii) T_2 (iii) T_{14} (iv) d, the common difference.

6. Find T_n of the arithmetic sequence 12, 15, 18, …
Hence write down the value of (i) T_{10} (ii) T_{40}.

7. In an arithmetic sequence $T_n = 5n - 1$.
Find T_1, T_2, and T_3.
Hence write down the value of a and the value of d.

8. In the sequence 4, 7, 10, …, write down the value of a and the value of d.
Find an expression for T_n and hence write down T_{20}.

9. The nth term of an arithmetic sequence is given by $T_n = 4n - 1$.
Write down the first three terms of this sequence.
Hence find the value of a and the value of d.

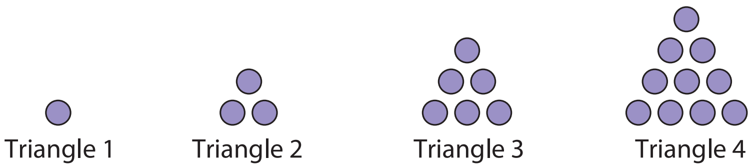
10. Find an expression for T_n of the arithmetic sequence

2, 6, 10, …

For what value of n is $T_n = 46$?

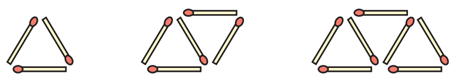
11. Find T_n of the arithmetic sequence 1, 3, 5, … .
For what value of n is $T_n = 87$?

12. These designs are made by arranging counters in triangles.



(i) Draw the 5th triangle.
(ii) How many counters are there in triangle 6?
(iii) Write down the number sequence formed by the first six triangles.
Explain why this is not an arithmetic sequence.

13. Complete the table of values for this sequence of matchstick patterns.



Number of triangles	1	2	3	4	5
Number of matchsticks	3				

(i) Explain why the sequence generated by the numbers of matchsticks is an arithmetic sequence.
(ii) Write an expression for the nth term of the sequence.
(iii) How many matchsticks are needed for the 30th term of the sequence?
(iv) Which term of the pattern has 81 matchsticks?

14. Find T_n of the arithmetic sequence 8, 5, 2, … .
For what value of n is $T_n = -34$?

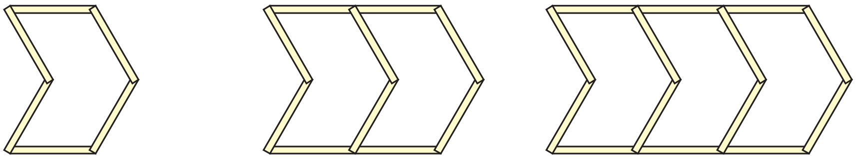
15. If T_n of the arithmetic sequence 4, 7, 10, 13, … is 127, find the value of n.

16. Which term of the arithmetic sequence 6, 11, 16, … is 186?

17. Which term of the arithmetic sequence $-8, -6, -4, \ldots$ is 38?

18. In the sequence 15, 20, 25, 30, …, $T_n = 215$. Find n.

19. Here is a pattern made from sticks.



Pattern number 1 Pattern number 2 Pattern number 3

(i) How many sticks are there in the 5th pattern?
(ii) Find, in terms of n, the number of sticks needed for the nth pattern.
(iii) How many sticks are needed for the 20th pattern?
(iv) Which pattern has 122 sticks?

20. T_n of a sequence is $T_n = n^2 + 4$.
Find T_1, T_2 and T_3 and hence state if the sequence is arithmetic.

Section 10.5 Finding the values of *a* and *d*

If we are given any two terms of an arithmetic progression, we can use simultaneous equations to find the values of a and d. We can then find any other term of the sequence.

Example 1

T_4 of an arithmetic sequence is 11 and $T_9 = 21$.
Find the values of a and d and hence find T_{50}.

$T_n = a + (n - 1)d$
$T_4 = 11 \Rightarrow a + 3d = 11 \ldots (1)$
$T_9 = 21 \Rightarrow \underline{a + 8d = 21} \ldots (2)$
Subtracting: $-5d = -10$
$\Rightarrow d = 2$

Substituting 2 for d in (1) we get
$a + 3(2) = 11$
$\Rightarrow a + 6 = 11 \Rightarrow a = 5$
$\therefore \; a = 5$ and $d = 2$

$T_{50} = a + 49d$
$= 5 + 49(2) = 5 + 98$
$= 103$

Example 2

If $x + 1$, $2x - 2$, and $2x + 1$ are three consecutive terms of an arithmetic sequence, find the value of x.
Hence write down T_n and T_{100} of the sequence.

If $x + 1, 2x - 2$ and $2x + 1$ are in arithmetic sequence, then

$$T_2 - T_1 = T_3 - T_2$$
$$\Rightarrow \quad (2x - 2) - (x + 1) = (2x + 1) - (2x - 2)$$
$$\Rightarrow \quad 2x - 2 - x - 1 = 2x + 1 - 2x + 2$$
$$\Rightarrow \quad x - 3 = 3$$
$$\Rightarrow \quad x = 6$$

The first three terms are: 7, 10, 13.

$\Rightarrow a = 7$ and $d = 3$

$$T_n = a + (n - 1)d$$
$$= 7 + (n - 1)3$$
$$= 7 + 3n - 3$$
$$\Rightarrow \quad T_n = 3n + 4$$
$$\Rightarrow \quad T_{100} = 3(100) + 4 = 304$$

Exercise 10.5

1. The first term of an arithmetic sequence is 5.
If the fifth term is 33, find d, the common difference.
Hence find T_n and T_{20}.

2. In an arithmetic sequence, $T_4 = 14$ and $T_9 = 34$.
Find the values of a and d and hence write down the value of T_{13}.

3. In an arithmetic sequence, $T_5 = 21$ and $T_{10} = 41$.
Find the values of a and d.
Hence find T_n and T_{60}.

4. In an arithmetic sequence, the eighth term is -18 and the third term is 12.
Find the values of a and d.
Hence find T_{100}.

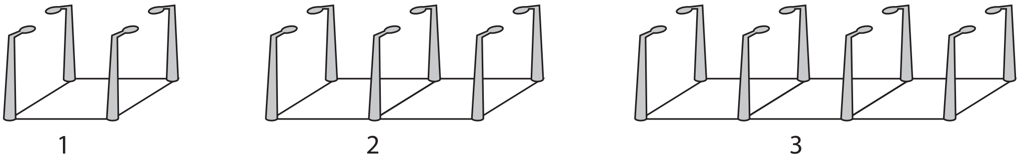
5. In an arithmetic sequence, $T_3 = 4$ and $T_{10} = -17$.
Find the values of a and d.
Write down T_n of the sequence and find the value of n for which $T_n = -47$.

6. In an arithmetic sequence, the first term is 3 and $T_6 = 2T_3$.
(i) Find the value of the common difference, d.
(ii) Find T_n, the nth term.

7. In an arithmetic sequence, $T_1 + T_5 = 0$ and $T_{13} = 20$.
(i) Find the value of a and the value of d.
(ii) Show that the seventh term is twice the fifth term.

8. In an arithmetic sequence, $T_4 = -9$ and $T_{15} = -31$.
Find the values of a and d.
Write down T_n of the sequence and hence find which term is equal to -81.

9. Lamp-posts are put at the end of every 100 m stretch of a motorway, as shown,



(i) How many lamp-posts are needed for 500 m of motorway?
(ii) Write down, as a number sequence, the number of lamp-posts required for 100 m, 200 m, 300 m, 400 m, … .
(iii) Find an expression in n for the nth term of this sequence.
(iv) Use the expression found in (iii) to write down the number of lamp-posts needed for 8 km of motorway.
(v) The M51 is a motorway being built.
The contractor has ordered 2402 lamp-posts.
How long is this motorway?

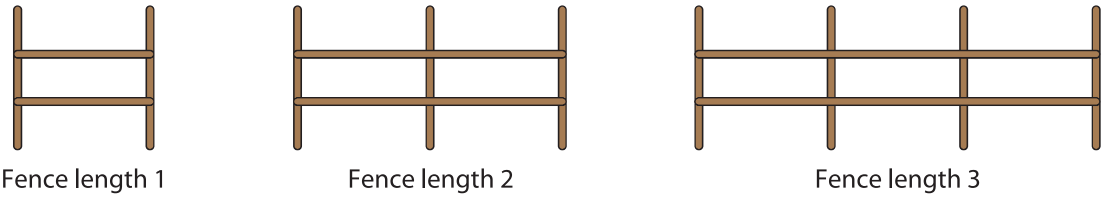
10. In an arithmetic sequence, $T_1 + T_3 = 12$ and $T_4 + T_6 = 24$.
Find the values of a and d.

11. In an arithmetic sequence, the sixth term is 20 and the tenth term is four times the second term.
Find the values of a and d. Hence calculate T_{100}.

12. If x, $2x + 3$ and $4x + 5$ form three consecutive terms of an arithmetic sequence, find the value of x.

13. Find the value of x in each of the following arithmetic sequences:
(i) $x - 1, x + 1, 3x - 3$
(ii) $x + 4, 3 - x, x + 10$.

14. Tommy builds fences in different lengths using pieces of wood.



(i) Sketch fence length 5.
Tommy counted how many pieces he needed to make each fence length.
He then drew up the table below.

Fence length	1	2	3	4	5	6
Number of pieces	4	7	10			

(ii) Complete the table to show how many pieces of wood he would use for fence lengths 4, 5 and 6.

(iii) Write down, in terms of n, an expression for the number of pieces of wood needed for fence length n.

(iv) How many pieces of wood are needed for fence length 40?

(v) If 91 pieces of wood are needed, what is the number of the fence length?

Section 10.6 Arithmetic series

When the terms of an arithmetic sequence are added, they form an **arithmetic series**.

For example, 1, 3, 5, 7, … is an arithmetic sequence
but $1 + 3 + 5 + 7 \ldots$ is an arithmetic series.

The $+$ sign between the terms changes a sequence to a series.

We use the notation $\boldsymbol{S_n}$ to denote the sum of the first n terms of a series.

Thus S_1 = sum of one term, i.e., $S_1 = T_1$

$S_2 = T_1 + T_2$

$S_3 = T_1 + T_2 + T_3$

The formula for S_n of an arithmetic series is given on the right.

The sum to n terms of an arithmetic series is given by the formula

$$S_n = \frac{n}{2}\{2a + (n - 1)d\}.$$

Example 1

Find S_n and hence S_{20} of the series $5 + 8 + 11 + 14 + \ldots$.

In the series $5 + 8 + 11 + 14 + \ldots$, $a = 5$ and $d = 3$.

$$S_n = \frac{n}{2}\{2a + (n - 1)d\}$$

$$= \frac{n}{2}\{2(5) + (n - 1)(3)\} \ldots (a = 5, d = 3)$$

$$= \frac{n}{2}\{10 + 3n - 3\}$$

$$S_n = \frac{n}{2}\{3n + 7\}$$

$$S_{20} = \frac{20}{2}\{60 + 7\}$$

$$= 10(67)$$

$$= 670$$

Example 2

Given the arithmetic series $5 + 7 + 9 + \ldots$.
If $S_n = 192$, find the value of n.

$$S_n = 192$$
$$\frac{n}{2}\{2a + (n - 1)d\} = 192$$
$$\frac{n}{2}\{10 + (n - 1)2\} = 192 \ldots \text{(}a = 5 \text{ and } d = 2\text{)}$$
$$\frac{n}{2}\{10 + 2n - 2\} = 192$$
$$\frac{n}{2}\{2n + 8\} = 192$$
$$\frac{2n^2}{2} + \frac{8n}{2} = 192$$
$$n^2 + 4n - 192 = 0$$
$$(n - 12)(n + 16) = 0$$
$$n = 12 \ldots \text{disregard the negative answer } -16$$
$$\therefore\ S_{12} = 192$$

Note: $S_4 = T_1 + T_2 + T_3 + T_4$
$S_5 = T_1 + T_2 + T_3 + T_4 + T_5$

Subtracting: $S_5 - S_4 = T_5$

Similarly $S_n - S_{n-1} = T_n$

Remember:
$T_n = S_n - S_{n-1}$

Example 3

In an arithmetic series, $S_n = n^2 + 2n$.
Find S_1, S_2 and S_3 and hence write down T_1, T_2 and T_3.

$$S_n = n^2 + 2n \Rightarrow S_1 = 1^2 + 2(1) = 3 \Rightarrow T_1 = 3$$
$$S_2 = 2^2 + 2(2) = 8 \Rightarrow T_1 + T_2 = 8 \Rightarrow T_2 = 5$$
$$S_3 = 3^2 + 2(3) = 15 \Rightarrow T_1 + T_2 + T_3 = 15$$
$$\Rightarrow 3 + 5 + T_3 = 15$$
$$\Rightarrow T_3 = 7$$

$\therefore$ the first three terms are 3, 5, 7.

Exercise 10.6

1. For the arithmetic series $2 + 5 + 8 + \ldots$,
(i) find the value of a and d
(ii) find the sum of the first 12 terms.

2. Find the sum of the first 20 terms of the series
$3 + 7 + 11 + 15 + \ldots$

3. Find S_n and hence S_{16} of the arithmetic series
$1 + 4 + 7 + 10 + \ldots$

4. The first four terms of a series are $7 + 10 + 13 + 16 + \ldots$
Find S_8, the sum of the first eight terms.

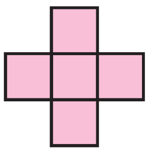
5. Write down the value of a and the value of d for the series
$16 + 12 + 8 + 4 + \ldots$
Hence find S_{24} of the series.

6. In an arithmetic series the nth term, $T_n = 5n - 2$.
Find the values of a and d and hence find S_{16} of the series.

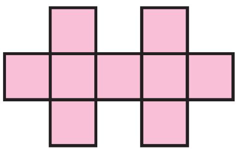
7. Show that S_n of the series $1 + 2 + 3 + \ldots$ is $\frac{n}{2}(n + 1)$.
Hence find the sum of the series $1 + 2 + 3 + \ldots + 100$.

8. S_n of the series $-4 - 2 + 0 + 2 + \ldots$ is 84.
(i) Write down the value of a and the value of d.
(ii) Find the value of n.

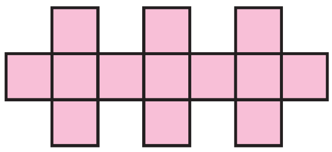
9. Here are some patterns made of squares.



Pattern number 1

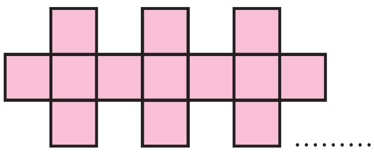


Pattern number 2



Pattern number 3

The diagram on the right shows part of Pattern number 4.



Pattern number 4

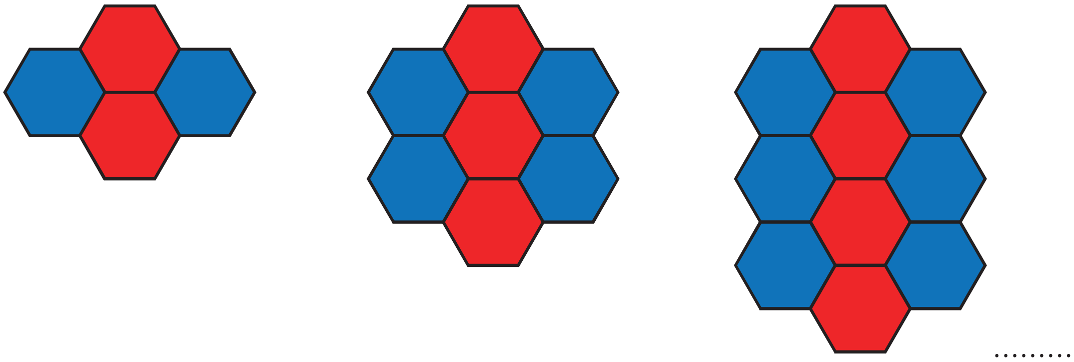
(i) Copy and complete Pattern number.
(ii) How many squares are there in Pattern 6?
(iii) Find an expression for the number of squares in Pattern n.
(iv) How many squares are there in total in the first 20 Patterns?

10. In an arithmetic series, $T_5 = 9$ and $T_8 = 27$.
(i) Find the values of a and d.
(ii) Find S_{10} of the series.

11. In an arithmetic series, $T_3 = 0$ and $T_8 = 10$.
Find the values of a and d and hence find S_n of the series.
How many terms of the series must be added so that their sum is 36?

12. Find S_n of the series $5 + 8 + 11 + 14 + \dots$.
If $S_n = 98$, find the value of n.

13. A student made tile designs using red and blue tiles, as shown.



(i) Find an expression in n for the total number of
  (a) red tiles used in the nth design
  (b) blue tiles used in the nth design.
(ii) Find, in terms of n, an expression for the total number of tiles used in the nth design.
(iii) How many tiles in total are needed to complete 10 designs using the same pattern?

14. Which term of the series $3 + 8 + 13 \dots$ is 98?
Now find the sum of these terms.

15. S_n of an arithmetic series is given by $S_n = n^2 + 6n$.
Find S_1 and S_2 and hence write down the values of T_1 and T_2.

16. (i) Write down the 10th term of the sequence which begins $3, 7, 11, 15, \dots$
(ii) Write down an expression for the nth term of this sequence.
(iii) Show that 1997 cannot be a term in this sequence.
(iv) Calculate the number of terms in the sequence $3, 7, 11, 15, \dots, 399$.
(v) Hence find the sum of the series $3 + 7 + 11 + 15 + \dots + 399$.

17. The nth term of an arithmetic series is $T_n = 52 - 4n$.
(i) Find the values of a and d.
(ii) Find which term is zero.
(iii) Find the sum of the terms which are positive.

18. The sum of the first n terms of an arithmetic series is given by

$$S_n = 4n^2 - 8n.$$

(i) Use S_1 and S_2 to find the first term and the common difference.
(ii) How many terms of the series must be added to give a sum of 252?

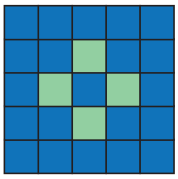
19. In an arithmetic series, $T_5 = 21$ and $T_{10} = 11$.

(i) Find the first term and the common difference.

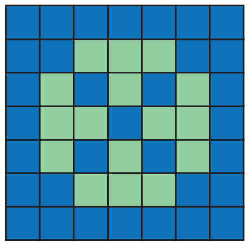
(ii) Find the sum of the first 20 terms.

(iii) For what value of n is $S_n = 0$?

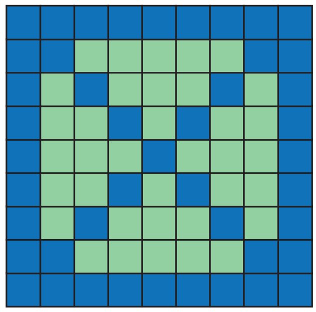
20. The first three patterns of a tiling sequence are shown below.
The sequence continues in the same way.



1st pattern



2nd pattern



3rd pattern

In each pattern, the tiles form a square of blue and green tiles.

(i) In the table below, write down the number of blue tiles needed for each of the first five patterns.

Pattern	1	2	3	4	5
Number of blue tiles	21	33			

(ii) Find, in terms of n, an expression for the number of blue tiles needed for the nth pattern.

(iii) Use the formula for T_n, found in (ii) above, to find the number of blue tiles in the 10th pattern.

(iv) Find, in terms of n, a formula for the total number of blue tiles in the first n patterns.

(v) How many patterns can be made with 399 blue tiles?

Section 10.7 Quadratic sequences

1, 4, 9, 16, 25, … is the sequence of square numbers.
Since $T_1 = 1^2, T_2 = 2^2, T_3 = 3^2, \dots T_n = n^2$.

Sequences that have an nth term containing n^2 as the highest power are called quadratic sequences.

Let us examine the first seven terms of the sequence $T_n = n^2$.

1 4 9 16 25 36 49

3 5 7 9 11 13 first difference

2 2 2 2 2 second difference

Notice that the second differences are all the same, i.e., 2.

Now let us look at the sequence with nth term $= 2n^2 - n$.
The first five terms of this sequence are

1		6		15		28		45	
	5		9		13		17		first difference
		4		4		4			second difference

Here the second differences are all the same, i.e., 4.

Notice that in each of the quadratic sequences above the coefficient of n^2 in the nth term is half the second difference.

Remember

In a quadratic sequence, the coefficient of n^2 in the nth term is half the second difference.

Investigation

Write out the first five terms of the sequence with $T_n = 2n^2 + 3$.
Write out the first differences and the second differences between consecutive terms.
Investigate if the coefficient of n^2 in the nth term is half the value of the second difference.

Finding the *n*th term of a quadratic sequence

The nth term of a quadratic sequence will always be of the form

$$T_n = an^2 + bn + c$$

We now use a difference table to find the values of a, b and c as shown in the following example.

Example 1

Find the nth term of the sequence 3, 10, 21, 36

The terms and the 1st and 2nd differences are shown on the right.

Terms	3		10		21		36
1st differences		7		11		15	
2nd differences			4		4		

T_n will be of the form $T_n = an^2 + bn + c$
$a = 2$ … half the value of the second difference

$\therefore\ T_n = 2n^2 + bn + c$

We now express T_1 and T_2 in terms of b and c.

$T_n = 2n^2 + bn + c \Rightarrow T_1 = 2 + b + c$

But $\quad T_1 = 3 \Rightarrow 2 + b + c = 3 \Rightarrow b + c = 1 \ldots$ 1

$T_2 = 8 + 2b + c$

But $\quad T_2 = 10 \Rightarrow 8 + 2b + c = 10 \Rightarrow 2b + c = 2 \ldots$ 2

We now solve the simultaneous equations 1 and 2

$b + c = 1 \ldots$ 1	$b + c = 1$
$2b + c = 2 \ldots$ 2	$\Rightarrow 1 + c = 1$
$-b = -1 \Rightarrow b = 1$	$\Rightarrow c = 0$

$\therefore\; T_n = 2n^2 + n \ldots$ $a = 2, b = 1, c = 0$

Exercise 10.7

1. Find the next two terms of these quadratic sequences by finding the first and second differences:

(i) 3, 4, 6, 9, 13, … (ii) 3, 6, 11, 18, 27, … (iii) 2, 7, 14, 23, 34, …

2. Which of these sequences are quadratic?

(i) 6, 8, 12, 18, 26, 36, … (ii) 6, 8, 10, 12, 14, 16, …
(iii) 3, 4, 7, 12, 19, 28, … (iv) 0, 3, 8, 15, 24, …

3. Find the first 5 terms of the sequences with these nth terms:

(i) $T_n = n^2 + 4$ (ii) $T_n = n^2 - 1$ (iii) $T_n = 2n^2 + n + 1$

4. Find the 10th term of the sequence with $T_n = n^2 + 2n - 4$.

5. Write the sequence 4, 7, 12, 19, 28, … as follows

4 7 12 19 28

_ _ _ _ first difference

_ _ _ second difference

If $T_n = an^2 + bn + c$, use the second difference to write down the value of a.

6. Find an expression for the nth term of each of these quadratic sequences:

(i) 5, 8, 13, 20, 29, … (ii) 2, 8, 18, 32, 50, …

7. Find an expression for the nth term of the sequence

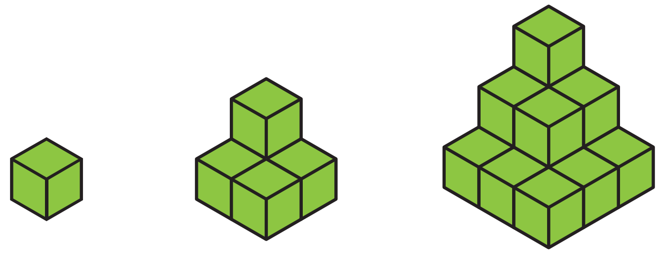
7, 10, 15, 22, 31, …

8. Show that the nth term of the quadratic sequence 8, 15, 26, 41, 60, … is $2n^2 + n + 5$.

9. Use a difference table to work out the nth term of this sequence:

3, 8, 15, 24, 35, …

10. Each layer of cubes in these designs is a square.



Model 1 Model 2 Model 3

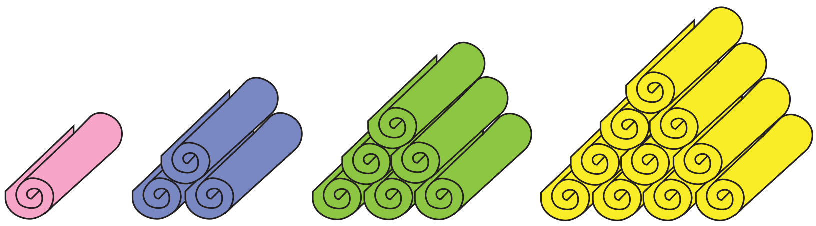
(i) How many cubes will be in Model 4?

(ii) Use the pattern to write down the number of cubes in Model 5.

(iii) Show that the expression $\frac{n}{6}(n + 1)(2n + 1)$ gives the correct number of cubes in Model 3 and Model 4.

(iv) Use the expression for the nth term to find how many cubes in Model 10.

11. Here are some rugs stacked in a carpet showroom.



Stack 1 Stack 2 Stack 3 Stack 4

(i) Copy and complete the table on the right.

Stack number	1	2	3	4	5	…
Number of rugs	1	3				…

(ii) Draw a difference table for the sequence for the number of rugs.

(iii) Use the differences to find an expression for the nth term of this sequence.

(iv) Use the nth term to find the number of rugs in Stack 20.

Test yourself 10

1. (i) The first three terms of an arithmetic sequence are $5, 8, 11, \ldots$

(a) Write down the first term and the common difference.

(b) Find an expression in n for the nth term of the sequence.

(c) Which term of the sequence is 62?

(ii) In an arithmetic sequence, $T_3 = 11$ and $T_7 = 27$.

(a) Find the value of the first term and the common difference.

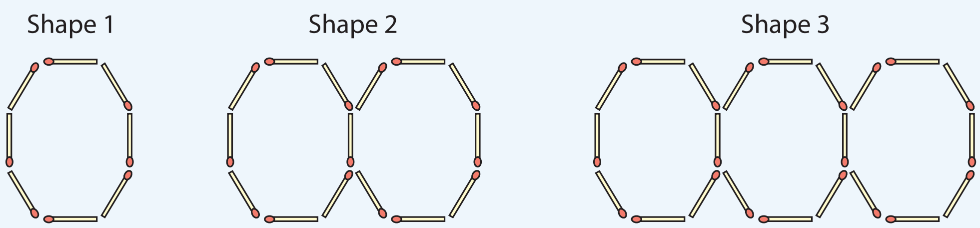
(b) Find the sum of the first 10 terms of the series.

2. (i) The first two terms of an arithmetic sequence are $5, 0, \ldots$.

(a) Find d, the common difference.

(b) Find T_n and hence T_{10} of the sequence.

(ii) Look at these shapes made with sticks.



(a) Copy and complete this table.

Shape number	1	2	3	4	5
Number of sticks	8				

(b) Find an expression in n for the number of sticks in the nth shape.

(c) How many sticks are there in shape 12?

(d) Find the sum of all the sticks used in the first 20 shapes.

3. (i) An arithmetic sequence is $8, 6, 4, \ldots$.

(a) Write down the value of a, the first term and d, the common difference.

(b) Find an expression in n for the nth term of the sequence.

(c) Which term of the sequence is -20?

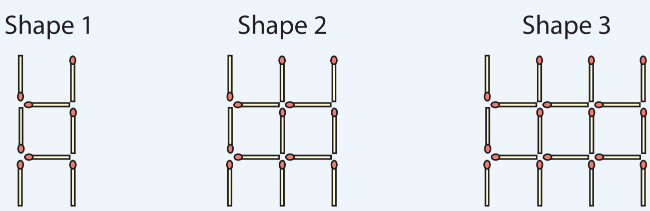
(ii) Write down the formula for S_n, the sum to n terms of an arithmetic series.
The common difference of the series is 3 and $S_8 = 132$.
Find the first 3 terms of the series and hence find the value of T_{24}.

4. (i) The nth term of an arithmetic sequence is given by $T_n = 12 - 4n$.

(a) Find the first term and the common difference.

(b) Which term of the sequence is -64?

(ii) A pattern of shapes is shown below.



(a) Copy and complete this table for the shapes above.

Shape	1	2	3	4	5
Number of matches	8				

(b) Find an expression in n for the nth term of the sequence formed by the numbers of matches required for successive patterns.

(c) How many matches would there be in Shape 20?

(d) Find the sum of all the matches used in the first 12 shapes.

5. (i) Which *one* of the following sequences is arithmetic?

(a) $6, 4, 0, \ldots$ (b) $3, -1, 3, -\ldots$ (c) $-5, -3, -1, 1, \ldots$

Now find T_n of this arithmetic sequence.

(ii) Write down the first differences and second difference between the terms of the quadratic sequence

$8, 15, 26, 41, 60, \ldots$

Hence find an expression in n for the nth term of the sequence.

6. (i) In an arithmetic sequence, the first term is 9 and the common difference is 4. Find T_n, the nth term, and hence write down the value of T_{10}.

(ii) In an arithmetic series $T_{10} = 19$ and $S_{10} = 55$.

(a) Write down the formula for T_n and S_n of an arithmetic series.

(b) Find the first term and common difference of the sequence above.

7. (i) The first three terms of an arithmetic series are $2 + 8 + 14 \ldots$

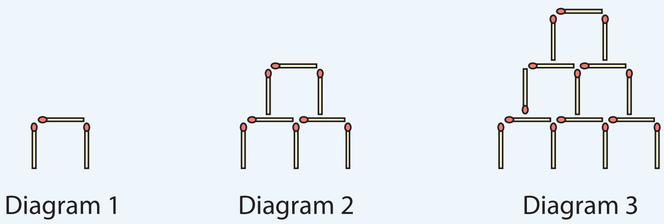
(a) Find d, the common difference.

(b) Find T_n, the nth term of the series.

(c) Find the value of n such that $T_n = 200$.

(d) Find the sum of the first 20 terms.

(ii) Here are the first 3 diagrams of a matchstick pattern.



(a) How many matchsticks will be in Diagram 4?

(b) Using the sequence formed, write down the number of matchsticks in Diagram 5.

(c) Explain why the sequence is quadratic.

(d) Find an expression for the nth term of the sequence.

(e) Use the nth term to find the number of matchsticks in Diagram 10.

Summary of key points...

1. Generating a sequence when given T_n

If you know the formula for T_n, the nth term of a sequence, you can find any term of the sequence by substituting a value for n.
$n = 1, 2, 3, \dots$ gives the first three terms.

2. Finding the *n*th term

For the sequence $5, 8, 11, \dots$ we find the nth term as follows

(i) The common difference between the terms is 3, so the nth term will be $3n \pm$ a number.

(ii) To find this number, we find what we add to 3 to get the first term 5. This number is 2.
$\therefore\ T_n = 3n + 2$.

3. Arithmetic sequences

For the arithmetic sequence $a, a + d, a + 2d, a + 3d, \dots$

$$T_n = a + (n - 1)d \qquad S_n = \frac{n}{2}\{2a + (n - 1)d\}$$

In an arithmetic sequence, the common difference d is

$\boldsymbol{d}$ **= any term − previous term**

4. Finding T_n from consecutive sums

$S_6 - S_5 = T_6$

In general $\boldsymbol{T_n = S_n - S_{n-1}}$.

5. Quadratic sequences

The nth term of a quadratic sequence will contain a term in n^2.
The coefficient of n^2 is half the value of the second difference.