Trigonometry

The trigonometry portion of our course can be divided into six sections:

* the basic definitions of angles and angle measure, trig ratios and trig functions for all angles,
* constructing and interpreting trig graphs,
* using practical trigonometry, e.g. sine rule, cosine rule, area of a sector, Pythagoras' theorem, to solve triangles, especially in 3D,
* proving the trig identities specified on the syllabus,
* using the 24 trig identities on the course to prove unseen identities and evaluate expressions,
* solving simple trig equations, being able to write down expressions for all solutions.

It should also be remembered that ideas and results from synthetic geometry are often required when answering questions on trigonometry.

In preparation for the trig questions, it is important to become familiar with all the formulae on pages 9 to 16 in the 'Formulae and Tables'. This is not to suggest that you should learn the formulae and special values, but at least you should be familiar with what is where. You should also learn to recognise expansions, e.g. should you meet

$$2\sin A\cos A$$

you should be able to look up that this is $\sin 2A$.

Being realistic, it is likely that the one 25 mark question on trig will probably deal with the more abstract areas of trig, e.g. trig graphs, proving trig identities. The more practical aspects will probably appear as major parts of one or two of the long questions in Section B.

This year, the trigonometric graphs element of the course has been expanded. This may mean that a question on trig graphs is more likely to appear this year.

1. Basic definitions

e.g. if $\tan A = \frac{t}{2}$, for $0° \le A \le 90°$,

express $\cos A$ and $\sin A$ in terms of t

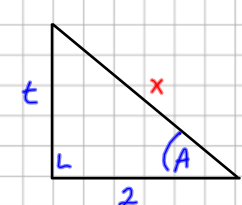
Sol

Soln CAH TOA

Pythagoras

$a^2 = b^2 + c^2$

sketch



Pythagoras

$x = \sqrt{t^2 + 2^2}$

$x = \sqrt{t^2 + 4}$

$\sin A = \frac{t}{\sqrt{t^2+4}}$

$\cos A = \frac{2}{\sqrt{t^2+4}}$

2. Trig graphs

e.g. A trigonometric function is given by

$$f(x) = a + b\cos cx\,,$$

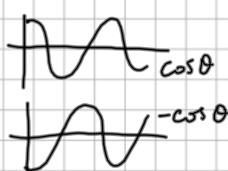
where $a, b, c \in \mathbb{R}$ and x is in degrees.

The range of the graph $y = f(x)$ is $[-10, 50]$ and its period is $72°$.

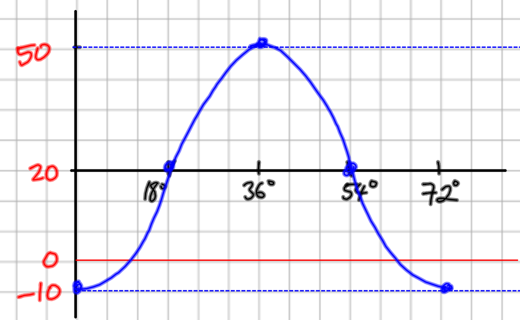
If $b < 0$, find the values of the constants a, b and c.

b is negative
⇒ negative Cos θ shape

Shape



Check: with calculator
$f(x) = 20 - 30\cos 5x$
Table: Start 0, end 72
Steps 18



The wave is shifted upwards by 20 ⇒ $a = 20$

Amplitude = 30
⇒ $|b| = 30(1) = 30$
but $b < 0$ ⇒ $b = -30$

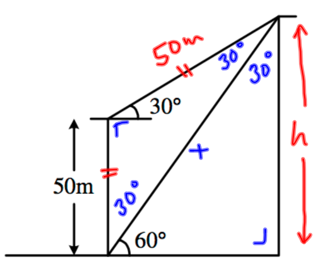
Period $= 72° = \frac{360°}{5}$

⇒ Frequency = 5 times frequency of cos θ
⇒ $c = 5$

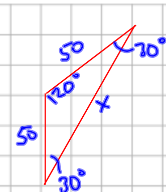
⇒ $f(x) = 20 - 30\cos 5x$

3. Right-angled triangles

e.g. A vertical tower and a vertical column are situated on horizontal ground.



From the foot of the tower, the angle of elevation of the top of the column is 60°; from the top of the tower, which is 50 m high, the angle of elevation is 30°. Find the height of the column.

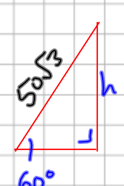


Sine Rule

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{x}{\sin 120°} = \frac{50}{\sin 30°}$$

$$x = \frac{50\sin 120°}{\sin 30°} = 50\sqrt{3}$$



SOH

$$\sin 60° = \frac{h}{50\sqrt{3}}$$

$$\Rightarrow h = 50\sqrt{3}\sin 60°$$

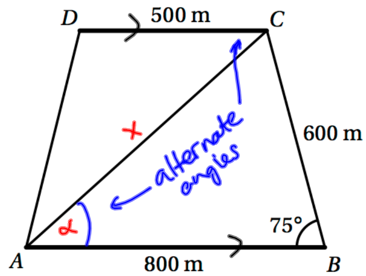
$$= 50\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)$$

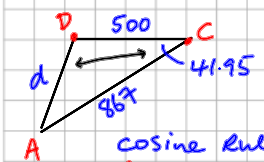
$$= 75\text{m}$$

Trigonometry Revision

4. Solving triangles

e.g. In the diagram, $[AB]$ and $[DC]$ are two parallel roads, where $|AB| = 800$ m and $|DC| = 500$ m. By measurement, it is determined that $|\angle ABC| = 75°$ and that $|BC| = 600$ m.



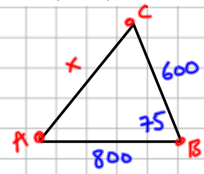


cosine Rule

$a^2 = b^2 + c^2 - 2bc \cos A$

Find

(i) $|AC|$, correct to the nearest metre, $(x = ?)$



Cosine Rule

$a^2 = b^2 + c^2 - 2bc \cos A$

$x^2 = (600)^2 + (800)^2 - 2(600)(800)\cos 75°$

$x^2 \approx 751533 \quad \Rightarrow x \approx 867$ m

(ii) $|\angle BAC|$, in degrees to two decimal places, $(\alpha = ?)$

Sine Rule $\qquad \dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

$\dfrac{\sin \alpha}{600} = \dfrac{\sin 75°}{867}$

$\Rightarrow \alpha = \sin^{-1}\left(\dfrac{600 \sin 75°}{867}\right) = 41.95°$

(iii) $|AD|$, correct to the nearest metre. $(d = ?)$

$d = \sqrt{[(500)^2 + (867)^2 - 2(500)(867)\cos 41.95°]}$

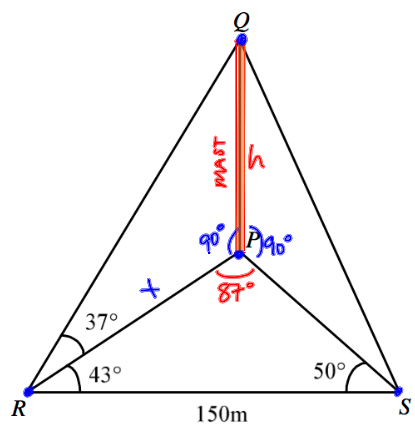
≈ 597 m

5. 3D problems

e.g. $[PQ]$ is a vertical mast and P, R, S are points on horizontal ground.

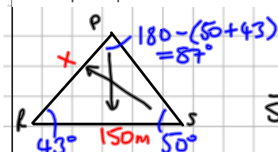
$|\angle PRS| = 43°$, $|\angle PSR| = 50°$,

$|\angle PRQ| = 37°$ and $|RS| = 150$ m.



Calculate

(i) $|PR|$, correct to one decimal place, $(x = ?)$



Sine Rule

$\dfrac{x}{\sin 50°} = \dfrac{150}{\sin 87°}$

$x = \dfrac{150 \sin 50°}{\sin 87°} \approx 115.1$ m

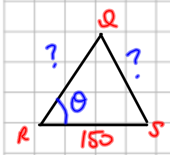
(ii) $|PQ|$, correct to one decimal place, $(h = ?)$

TOA

$\tan 37° = \dfrac{h}{115.1}$

$\Rightarrow h = 115.1 \tan 37° \approx 86.7$ m

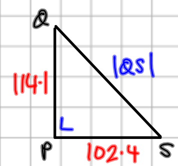
Plan: find sides and use the Cosine Rule



① $|RS| = ?$

② $|QS| = ?$

need $|PS|$



Sine Rule

$\theta = ?$

③ $|\angle QRS| = ?$

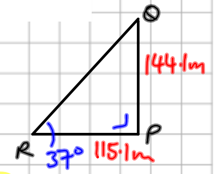
Cosine Rule

$a^2 = b^2 + c^2 - 2bc \cos A$

$\Rightarrow A = \cos^{-1}\left(\dfrac{a^2 - b^2 - c^2}{-2bc}\right)$

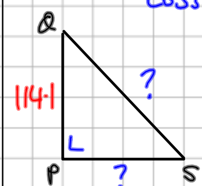
(iii) $|\angle RQS|$, correct to the nearest degree. $(\theta = ?)$

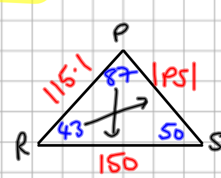
$|RQ| = ?$ CAH



$\cos 37° = \dfrac{115 \cdot 1}{|RQ|}$

$|RQ| = \dfrac{115 \cdot 1}{\cos 37°} \approx 144 \cdot 1\,m$





$|PS| = \dfrac{150 \sin 43°}{\sin 87°} \approx 102 \cdot 4\,m$

Pythagoras

$|QS| = \sqrt{(114 \cdot 1)^2 + (102 \cdot 4)^2} \approx 153 \cdot 3\,m$

$|\angle QRS| = \cos^{-1}\left(\dfrac{(153 \cdot 3)^2 - (150)^2 - (144 \cdot 1)^2}{-2(150)(144 \cdot 1)}\right)$

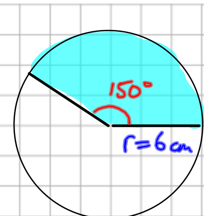
$\approx 63°$

6. Arcs and sectors

e.g. a sector of a circle of radius length 6cm has central angle 150°, express in terms of π the area of the sector

$A_{circle} = \pi r^2$

$A_{sector} = \dfrac{\theta}{360°} \pi r^2$



Sector Area?

$A_{sector} = \dfrac{150°}{360°} \pi (6)^2$

$= \dfrac{5}{12} \pi (36)$

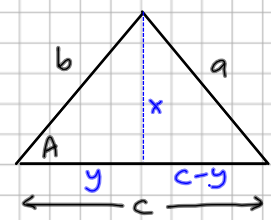
$= 15\pi$

7. Trig proofs

e.g. In a triangle, the sides a, b and c are opposite the angles A, B and C respectively. Prove that

$$a^2 = b^2 + c^2 - 2bc\cos A\,.$$

Prove cosine Rule



CAH

$\cos A = \frac{y}{b} \Rightarrow y = b\cos A$ ①

Pythagoras

$b^2 = x^2 + y^2$ ②

$a^2 = x^2 + (c-y)^2$

$a^2 = x^2 + c^2 - 2cy + y^2$

$a^2 = x^2 + y^2 + c^2 - 2cy$ ③

① + ② → ③

$\Rightarrow a^2 = b^2 + c^2 - 2bc\cos A$

8. Trig identities

e.g. Write $\tan 2A$ in terms of $\tan A$. Hence, or otherwise, find $\tan A$, where A is an acute angle, if $\tan 2A = \frac{3}{4}$. Do not use a calculator.

Double angle formulae

$\cos 2A = \cos^2 A - \sin^2 A$

$\sin 2A = 2\sin A\cos A$

$\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

In Log Tables

$\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

$\tan 2A = \frac{3}{4}$ $\quad \tan A = ?$

$\frac{3}{4} = \dfrac{2\tan A}{1-\tan^2 A}$

let $\tan A = t$

$\frac{3}{4} = \dfrac{2t}{1-t^2}$

$3(1-t^2) = 4(2t)$

$3 - 3t^2 = 8t$

$3t^2 + 8t - 3 = 0$

$(3t - 1)(t + 3) = 0$

$\Rightarrow 3t - 1 = 0$

$3t = 1$

$t = 1/3$ $\quad | \quad t = -3$

$\Rightarrow \tan A = \frac{1}{3}$ ✓ or $\tan A = -3$ ✗ reject

A is acute . ie. $A < 90°$

$\Rightarrow$ A is in 1st quadrant where $\tan A > 0$

$\Rightarrow A = \tan^{-1}(\frac{1}{3})$ *

* can't go further without a calculator

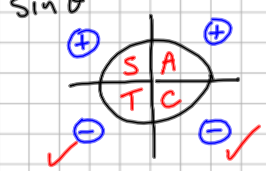
9. Trig equations

e.g. Find the general solution of the equation

$$\sin 2x = -\frac{\sqrt{3}}{2}$$

and use it to find all the solutions for $0° \le x \le 720°$.

$\sin\theta$



$1440° = 4(360°)$

$\Rightarrow$ up to 4 revolutions to be considered.

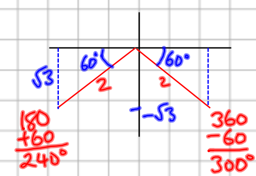
But $x = \theta/2$

$\sin 2x = -\frac{\sqrt{3}}{2}$

let $2x = \theta$ $\Rightarrow x = \theta/2$

$\Rightarrow 0° \le \theta/2 \le 720° \Rightarrow 0° \le \theta \le 1440°$

$\sin\theta = -\frac{\sqrt{3}}{2} \Rightarrow \theta = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$



between 0° and 360°

$\Rightarrow \theta = 240°$

or $\theta_2 = 300°$

Values for θ up to 1440°

$\theta_1 = 240°$, $\theta_2 = 300°$

$\theta_3 = 240 + 360 = 600°$, $\theta_4 = 300 + 360 = 660°$

$\theta_5 = 600 + 360 = 960°$, $\theta_5 = 660 + 360 = 1020°$

$\theta_6 = 960 + 360 = 1320°$, $\theta_6 = 1020 + 360 = 1380$

Values of x up to 720°

$x_1 = 240/2 = 120°$, $x_2 = 300/2 = 150°$

$x_3 = 600/2 = 300$, $x_4 = 660/2 = 330°$

$x_5 = 960/2 = 480°$, $x_5 = 1020/2 = 510°$

$x_6 = 1320/2 = 660°$, $x_6 = 1380/2 = 690°$