12. How many 4-digit numbers can be formed using the digits 4, 5, 6 and 7, if no digit can be repeated in a number?
(i) How many of these numbers are greater than 6000?
(ii) Find the probability that the number is greater than 6000?

13. Two women, *A* and *B*, and two men, *C* and *D*, sit in a row for a photograph.
(i) How many different arrangements of the four people are possible?
(ii) Write out the four possible arrangements that have the two women in the middle.
(iii) If an arrangement of the four people is chosen at random from all of the possible arrangements, what is the probability that the two women will be in the middle?

14. Shauna has five counters and she places them in a straight line.
They are of five different colours: red, white, green, blue and yellow.
(i) How many different arrangements are possible?
(ii) In how many arrangements is the first counter blue?
(iii) In how many arrangements is the first counter blue and the fifth counter green?

15. Evaluate each of the following:
(i) $5!$ (ii) $7!$ (iii) $\frac{6!}{3!}$ (iv) $4! \times 3!$

16. (i) Is $7! = 4! \times 3!$? (ii) Is $8! = 5! + 3!$?

17. Express $8!$ in the form
(i) $p(7!)$ (ii) $q(6!)$

18. If $10! + 9! = k(9!)$, find the value of k.

Section 6.12 Combinations

A selection of objects from a given set, without regard to the order, is called a **combination**.

If we take the letters *A*, *B*, *C*, *D* and select different groups of two letters, we have:

AB, *AC*, *AD*, *BC*, *BD*, *CD*, i.e. 6 groups

The group *AB* is the same as *BA* in this case as order does not matter.

We use the notation $\binom{4}{2}$ to denote the number of ways 2 objects can be selected from 4 objects.

$$\binom{4}{2} = \frac{4 \times 3}{2 \times 1}$$
… start at 4 and go down two terms
… 2! i.e. 2×1

Similarly $\binom{6}{3} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = \frac{120}{6} = 20$ and $\binom{8}{4} = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 70$

$\binom{8}{4}$ represents the number of ways 4 items can be chosen from 8 items.

In general $\binom{n}{r}$ is the number of ways in which r objects can be selected from n objects.

$\binom{n}{r}$ is pronounced 'n-c-r' or 'n choose r'.

Another word for **combination** is 'selection'.

$\binom{n}{r}$ may also be written as ${}^nC_r$ or nCr.

In electronic calculators it is written as nCr.

$\binom{n}{r} = \frac{n(n-1)\ \ldots}{r!\ \ldots}$ start at n and go down r terms; r factorial

To find the value of $\binom{8}{4}$ on a calculator, key in

8 [nCr] 4 [=]

Again $\binom{8}{5} = \frac{8 \times 7 \times 6 \times 5 \times 4}{5 \times 4 \times 3 \times 2 \times 1} = 56$

and $\binom{8}{3} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56.$

This shows that $\binom{8}{5} = \binom{8}{3}$ i.e. $\binom{8}{5} = \binom{8}{8-5}$

In general

$\binom{n}{r} = \binom{n}{n-r}$

Using the rule above, $\binom{10}{8} = \binom{10}{10-8} = \binom{10}{2} = \frac{10 \times 9}{2 \times 1} = 45.$

In practice, it is much shorter to work out $\binom{10}{2}$ than $\binom{10}{8}$.

$\binom{n}{n}$ and $\binom{n}{0}$

$\binom{8}{8} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 1$ and $\binom{8}{8} = \binom{8}{8-8} = \binom{8}{0} = 1$

These two examples illustrate that:

$\binom{n}{n} = 1$

$\binom{n}{0} = 1$

Example 1

How many different teams of 11 players can be selected from a panel of 14 players?

If the panel of 14 players includes the captain and one goalkeeper, how many teams can be selected if

(i) the goalkeeper has to be included

(ii) the goalkeeper and captain both have to be included?

The number of ways in which a team of 11 can be selected from 14 is $\binom{14}{11}$.

$\binom{14}{11} = \binom{14}{3} = \frac{14 \times 13 \times 12}{3 \times 2 \times 1} = 364$

(i) If the goalkeeper has to be included, then 10 players are selected from 13.

This can be done in $\binom{13}{10}$ ways.

$$\binom{13}{10} = \binom{13}{3} = \frac{13 \times 12 \times 11}{3 \times 2 \times 1} = 286$$

$\therefore$ 286 teams can be selected

(ii) If the goalkeeper and captain are included, then 9 players are selected from 12.

This can be done in $\binom{12}{9}$ ways.

$$\binom{12}{9} = \binom{12}{3} = \frac{12 \times 11 \times 10}{3 \times 2 \times 1} = 220$$

$\therefore$ 220 teams can be selected

Selecting from two different groups

There are 6 teachers and 8 students on a school council.
They want to select a delegation of 3 teachers and 3 students to meet the school principal.
In how many ways can this be done?
Here we have to select 3 teachers from 6 **AND** 3 students from 8.

3 teachers can be selected from 6 teachers in $\binom{6}{3}$ ways

3 students can be selected from 8 students in $\binom{8}{3}$ ways

Based on the **Fundamental Principle of Counting**, the word **AND** implies that the results of the two operations are **multiplied**.

AND $\Rightarrow$ Multiplication
OR $\Rightarrow$ Addition

Therefore the number of ways that three teachers and three students can be selected is

$$\binom{6}{3} \times \binom{8}{3}$$

$$\binom{6}{3} \times \binom{8}{3} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} \times \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 20 \times 56 = 1120.$$

Example 2

A committee of 4 people is to be formed from a group of 7 men and 6 women.

(i) How many different committees can be formed?

(ii) On how many of these committees is there an equal number of men and women?

(i) Here we are selecting 4 people from 13.

This can be done in $\binom{13}{4}$ ways.

$$\binom{13}{4} = \frac{13 \times 12 \times 11 \times 10}{4 \times 3 \times 2 \times 1} = 715$$

$\therefore$ 715 committees can be formed.

(ii) If there is an equal number of men and women, the committee will consist of 2 men and 2 women.

We can select 2 men from 7 men and 2 women from 6 women in

$\binom{7}{2} \times \binom{6}{2}$ ways

$$\binom{7}{2} \times \binom{6}{2} = \frac{7 \times 6}{2 \times 1} \times \frac{6 \times 5}{2 \times 1} = 315$$

$\therefore$ 315 committees can be formed.

Finding the value for *n*

$\binom{8}{2} = \frac{8 \times 7}{2 \times 1} = 28.$ Similarly, $\binom{n}{2} = \frac{n(n-1)}{2 \times 1} = \frac{n(n-1)}{2}$

Example 3

If $\binom{n}{2} = 45$, find the value of n, for $n \in N$.

$$\binom{n}{2} = \frac{n(n-1)}{2 \times 1}$$ ... go down 2 terms ... 2!

$$\binom{n}{2} = 45 \quad \Rightarrow \quad \frac{n(n-1)}{2} = 45$$

$\Rightarrow \quad n^2 - n = 90$

$\Rightarrow \quad n^2 - n - 90 = 0$

$\Rightarrow \quad (n-10)(n+9) = 0$

$\Rightarrow \quad n = 10, n = -9$

$\Rightarrow \quad n = 10$, as $n \in N$.

Exercise 6.12

1. Evaluate each of the following:

   (i) $\binom{5}{2}$ (ii) $\binom{7}{3}$ (iii) $\binom{8}{4}$ (iv) $\binom{10}{3}$ (v) $\binom{10}{7}$ (vi) $\binom{16}{2}$

   Verify your answer by using a calculator.

2. Show that (i) $\binom{8}{3} + \binom{8}{2} = \binom{9}{3}$ (ii) $\binom{10}{3} = \frac{10!}{3!7!}$

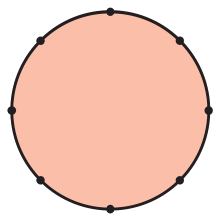
3. In how many ways can a board of 6 persons be selected from 10 persons?

4. In how many ways can two class representatives be selected from a class of 20 students?

5. A student has to select six subjects for the Leaving Certificate course from twelve subjects being offered in the school.
   (i) In how many ways can this be done?
   (ii) In how many ways can this be done if maths must be included in each selection?

6. A football team manager has 11 players from which to select a team of 7 players.
   (i) In how many ways can the team be chosen?
   (ii) If the 11 players include one goalkeeper, how many teams can be selected if the goalkeeper must be included in each team?

7. Eight points lie on a circle, as shown.
   How many different line segments can be drawn by joining any two of the eight points?



8. How many different subsets of four letters can be made from the set $\{a, b, c, d, e, f\}$?
   How many of these subsets contain the letter a?

9. In how many ways can a panel of 5 persons be selected from 6 men and 4 women?
   How many panels consist of
   (i) 3 men and 2 women (ii) 2 men and 3 women?

10. Seven persons A, B, C, D, E, F and G are eligible for selection on a team of five persons.
    (i) How many different teams can be selected?
    (ii) In how many of the teams will A be included?
    (iii) In how many of the teams will A and B be included?

11. A school council consists of 6 teachers and 5 pupils.
    (i) In how many ways can a committee of 4 be selected from this council?
    (ii) How many committees can be selected if each committee consists of 2 teachers and 2 pupils?

12. Seven people take part in a tennis competition.
How many matches will be played if each person must play each of the others?

13. A committee of 4 is to be chosen from 4 men and 5 women.
(i) How many committees can be chosen?
(ii) How many committees consist of 1 woman and 3 men?
(iii) How many committees consist of 2 women and 2 men?

14. (i) If $\binom{n}{2} = 28$, find n for $n \in N$. (ii) If $\binom{n}{2} = 55$, find n for $n \in N$.

15. A panel of 3 people is selected from a group of 15 doctors and 12 dentists.
In how many different ways can the 3 people be selected
(i) if there are no restrictions
(ii) if the selection must contain exactly 2 doctors?

16. There are 6 junior-cycle students and 5 senior-cycle students on the student council in a particular school.
A committee of 4 students is to be selected from the students on the council.
In how many different ways can the committee be selected if
(i) there are no restrictions
(ii) a particular student must be on the committee
(iii) the committee must consist of 2 junior-cycle students and 2 senior-cycle students?

A committee of 4 students is chosen at random.
(iv) Find the probability that all 4 students are junior-cycle students.