Differential Calculus

chapter 2

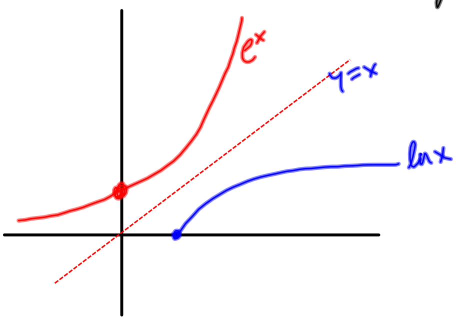
Section 2.9 Differentiating logarithmic functions

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$\ln x = \log_e x$

"natural log"



PROJECT MATHS

Text & Tests 7

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Logarithms to the base e are called **natural logarithms**.

Instead of using $\log_e x$, the abbreviation $\mathbf{\ln x}$ is generally used.

From our knowledge of logarithms, we know that the equation

$8 = 2^3$ can be written as $\log_2 8 = 3$.

Similarly, $x = e^y$ can be written as $\log_e x = y$ (or $\ln x = y$).

For this reason, $y = \ln x$ is the inverse of the function $y = e^x$.

To find the derivative of $y = \ln x$, we use the fact that if $y = \ln x$, then by definition $e^y = x$.

$$y = \ln x$$

$$\Rightarrow \quad x = e^y$$

$$\frac{dx}{dy} = e^y \quad \text{... differentiate with respect to } y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{e^y} \quad \text{... } \frac{dy}{dx} \text{ is the reciprocal of } \frac{dx}{dy}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x}$$

If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$.

When dealing with natural logarithms, the following results are important:

$\log_e e^x = x \log_e e = x$ $\qquad$ $\ln e^x = x$, for all $x \in R$

It should be further noted that

(i) $\log_e 1 = 0$ ... the log of 1 to any base is zero

(ii) $\log_e e = 1$ ... $(\log_k k = 1)$

The work involved in differentiating logarithmic functions can be simplified by using the laws of logarithms which are reproduced below:

Laws of Logarithms

(i) $\log_e(xy) = \log_e x + \log_e y$ $\qquad$ (ii) $\log_e\left(\frac{x}{y}\right) = \log_e x - \log_e y$

(iii) $\log_e x^n = n \log_e x$ $\qquad$ (iv) $\log_a x = \frac{\log_e x}{\log_e a}$

Note: If $y = \log_e(6x)$, we use logarithmic differentiation and the *Chain Rule* to find $\frac{dy}{dx}$.

$$y = \log_e(6x) \Rightarrow \frac{dy}{dx} = \frac{1}{6x} \cdot \frac{d}{dx}(6x) = \frac{1}{6x} \cdot \frac{6}{1} = \frac{1}{x}$$

In general, if $y = \log_e(f(x))$, $\frac{dy}{dx} = \frac{1}{f(x)} \cdot f'(x)$

INDICES

$$a^n \cdot a^m = a^{n+m}$$

$$\frac{a^n}{a^m} = a^{n-m}$$

$$(a^n)^m = a^{nm}$$

LOGS $\qquad$ P.21

$$\log n + \log m = \log(nm)$$

$$\log n - \log m = \log\left(\frac{n}{m}\right)$$

$$\log a^n = n \log a$$

$$\boxed{\begin{gathered} b^n = a \\ n = \log_{\boxed{b}} \boxed{a} \end{gathered}}$$

$$2^{?} = 5000$$

$$? = \log_2 5000 = 12.38$$

Example 1

Find $\frac{dy}{dx}$ if (i) $y = \log_e(4x^2 + 1)$ (ii) $y = \log_e(\sin^2 x)$.

If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$.

(i) Chain Rule

Outer: $\ln u$

inner: $4x^2+1$

$$y = \ln(4x^2+1)$$

$$\frac{dy}{dx} = \underbrace{\frac{1}{(4x^2+1)}}_{\text{Diff. outside}} \times \underbrace{(8x)}_{\text{Diff. inside}} = \frac{8x}{4x^2+1}$$

(ii) outer: $\ln u$

middle: u^2

inner: $\sin x$

$$\frac{1}{\tan x} = \cot x = \frac{\cos x}{\sin x}$$

P.13

$$y = \ln(\sin x)^2$$

$$\frac{dy}{dx} = \underbrace{\frac{1}{\sin^2 x}}_{\text{Diff. outside}} \times \underbrace{2(\sin x)^1}_{\text{Diff. middle}} \times \underbrace{\cos x}_{\text{Diff. inside}}$$

$$= \frac{2\cancel{\sin x}\cos x}{\sin^{\cancel{2}} x} = 2\cot x$$

Example 2

Given that $y = \log_e\left(\frac{1+x}{1-x}\right)$, show that $(1 - x^2)\frac{dy}{dx} = 2$.

Chain Rule and Quotient Rule

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Outside : ln (QUOTIENT)
inside : QUOTIENT

let $u = 1+x$

$\frac{du}{dx} = 1$

$v = 1-x$

$\frac{dv}{dx} = -1$

$$y = \ln\left(\frac{1+x}{1-x}\right)$$

$$\frac{dy}{dx} = \underbrace{\frac{1}{\left(\frac{1+x}{1-x}\right)}}_{\text{Diff. outside}} \cdot \underbrace{\left[\frac{(1-x)(1) - (1+x)(-1)}{(1-x)^2}\right]}_{\text{Diff inside}}$$

Flip!

$$= \left(\frac{\cancel{1-x}}{1+x}\right)\left(\frac{1 \cancel{-x} + 1 \cancel{+x}}{(1-x)^{\cancel{2}}}\right)$$

$$= \frac{2}{(1+x)(1-x)}$$

Exercise 2.9

Find $\frac{dy}{dx}$ of the functions in numbers (1–9):

1. $y = \log_e 5x$ **2.** $y = \log_e(2x + 3)$ **3.** $y = \log_e(3x^2)$