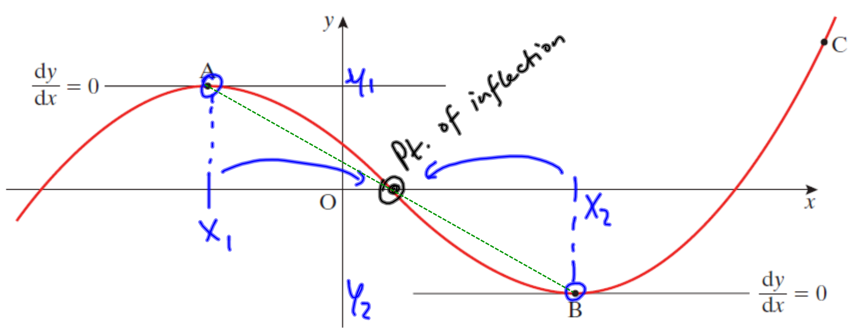
Chapter 3 Applications of Differential Calculus

Section 3.2 Stationary points

PROJECT MATHS Text & Tests 7

Points on a curve at which $\frac{dy}{dx}$ is zero are called **stationary points**.



In the curve above, A and B are stationary points and they are also **turning points**.

The turning point at A is called a **local maximum point**, since the value of the function at this point exceeds all values of the function immediately to the right or left of A.

A maximum value of a function is not necessarily the greatest value of the function.
This is illustrated in the curve above where the value of the function at C is greater than the value of the function at A.

The turning point at B above is called a **minimum turning point** or simply a **local minimum**.

The following steps should be used to find the turning points of a curve:

(i) find $\frac{dy}{dx}$ of the function

(ii) let $\frac{dy}{dx} = 0$ and solve the equation

(iii) for each value of x, find the corresponding value for y.

Example 1

Find the turning points of the curve $y = x + \frac{1}{x}$.

function in index form	$y = x + x^{-1}$
Slope	$\frac{dy}{dx} = 1 - x^{-2}$
$\frac{dy}{dx} = 0$	$\Rightarrow 1 - x^{-2} = 0$
	$1 - \frac{1}{x^2} = 0$
$\times$ LCD	$x^2 - 1 = 0$
DOTS	$(x+1)(x-1) = 0$
Sub into fn. $y = x + \frac{1}{x}$	$x = +1$ or $x = -1$
$x = 1$	$y = 1 + \frac{1}{1} = 2$ pt $(1, 2)$
$x = -1$	$y = -1 + \frac{1}{-1} = -2$ pt $(-1, -2)$

Example 4

Verify that the curve $y = \dfrac{x+2}{2x-3}$ has no turning points or points of inflection.

Verify also that the curve is decreasing for all values of $x \in R$.

Quotient Rule

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$u = x+2$

$\frac{du}{dx} = 1$

$v = 2x-3$

$\frac{dv}{dx} = 2$

at min/max $\frac{dy}{dx} = 0$

$(2x-3)^2 \geq 0$

If $\frac{dy}{dx} < 0 \Rightarrow$ decreasing

$$\Rightarrow \frac{dy}{dx} = \frac{(2x-3)(1) - (x+2)(2)}{(2x-3)^2}$$

$$= \frac{\cancel{2x} - 3 - \cancel{2x} - 4}{(2x-3)^2}$$

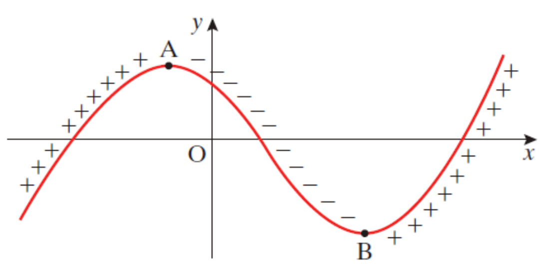
$$= \frac{-7}{(2x-3)^2}$$

Can $\frac{-7}{(2x-3)^2} = 0$? no

because $\frac{-7}{(2x-3)^2} < 0$

Determining the nature of a turning point

In the diagram below, the positive signs (+) indicate where the slope of the curve is positive $\left(\text{i.e. } \frac{dy}{dx} > 0\right)$, and the negative signs indicate where the slope of the curve is negative $\left(\text{i.e. } \frac{dy}{dx} < 0\right)$.



At a maximum turning point, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$, i.e. negative

At a minimum turning point, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$, i.e. positive

Example 2

Find the turning points of the curve $y = x^3 - 9x^2 + 15x$ and determine the nature of these turning points.

max or min?

at max/min
$\frac{dy}{dx} = 0 \Rightarrow$

$\frac{dy}{dx} = 3x^2 - 18x + 15 = 0$

$(3x-3)(x-5) = 0$

$3x - 3 = 0$ | $x = 5$

$x = 1$

Turning Points
$x=1$, $y=?$
$x=5$, $y=?$

$y = (1)^3 - 9(1)^2 + 15(1) = 7$ pt. $(1, 7)$

$y = (5)^3 - 9(5)^2 + 15(5) = -25$ pt $(5, -25)$

max/min?

at min $\Rightarrow \frac{d^2y}{dx^2} > 0$

at max $\Rightarrow \frac{d^2y}{dx^2} < 0$

$\frac{d^2y}{dx^2} = 6x - 18$

$\frac{d^2y}{dx^2}_{(x=1)} = 6(1) - 18 = -12 < 0$

$\Rightarrow (1, 7)$ is a local max.

$\frac{d^2y}{dx^2}_{(x=5)} = 6(5) - 18 = 30 - 18 = 12 > 0$

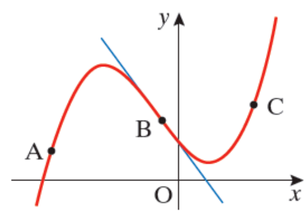
$\Rightarrow (5, -25)$ is a local min.

Points of Inflection

The curve traced in the diagram below is said to be **concave upwards** from the point A to the point B, and **concave downwards** from the point B to the point C.

The point B, where the curve changes from being concave upwards to concave downwards, is called a **point of inflection**.

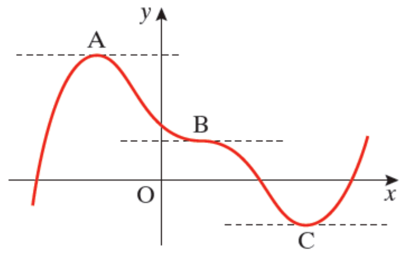
At a point of inflection, the tangent to the curve crosses the curve.



At a point of inflection, B, on the curve $y = f(x)$, $\frac{d^2y}{dx^2} = 0$ and changes sign as the curve passes through B.

Note: The point B on the given curve is a point of inflection. The tangent to the curve at B is also parallel to the x-axis.

The point B is called a **saddle point** or a **horizontal point of inflection.**



Thus, to find the point(s) of inflection of a curve:

(i) find $\frac{d^2y}{dx^2}$

(ii) solve the equation $\frac{d^2y}{dx^2} = 0$

(iii) for each value of x, find the corresponding value of y.

Example 3

Find the point of inflection of the curve $y = x^3 - 3x^2 - 2$.

at pt. of inflection

$\frac{d^2y}{dx^2} = 0$

$\frac{dy}{dx} = 3x^2 - 6x$

$\frac{d^2y}{dx^2} = 6x - 6$

$\Rightarrow$ $6x - 6 = 0 \Rightarrow x = 1$

$x = 1, y = ?$

$y = (1)^3 - 3(1)^2 - 2$

$y = 1 - 3 - 2 = -4$

$\Rightarrow$ pt. $(1, -4)$