chapter 4 Integration

Section 4.2 Integrating exponential and trigonometric functions

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In chapter 2, it was found that if

(i) $f(x) = e^x$, then $f'(x) = e^x \Rightarrow \int e^x\,dx = e^x + c$

(ii) $f(x) = e^{ax}$, then $f'(x) = ae^{ax} \Rightarrow \int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$

(iii) $f(x) = a^x$, then $f'(x) = a^x \ln a \Rightarrow \int a^x\,dx = \frac{a^x}{\ln a} + c$

Example 1

Find the antiderivative of each of the following:

(i) $\int e^{3x}\,dx$ (ii) $\int (e^{4x} + 6x)\,dx$ (iii) $\int (e^{5x} + 2)\,dx$ (iv) $\int (e^{x} + e^{-x})\,dx$

Integration

Constants of integration omitted.

$f(x)$	$\int f(x)dx$
$x^n, (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
* e^{ax}	$\frac{1}{a}e^{ax}$
$a^x \quad (a > 0)$	$\frac{a^x}{\ln a}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\tan x$	$\ln\lvert\sec x\rvert$
$\frac{1}{\sqrt{a^2 - x^2}} \quad (a > 0)$	$\sin^{-1}\frac{x}{a}$
$\frac{1}{x^2 + a^2} \quad (a > 0)$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$

(i) $\int e^{3x}dx = \frac{1}{3}e^{3x} + c$

(ii) $\int (e^{4x} + 6x)dx$

$= \frac{1}{4}e^{4x} + \frac{6x^2}{2} + c$

$= \frac{1}{4}e^{4x} + 3x^2 + c$

(iii) $\int (e^{5x} + 2)dx$

$= \frac{1}{5}e^{5x} + 2x + c$

(iv) $\int (e^{x} + e^{-x})dx$

$= e^x + \frac{1}{-1}e^{-x} + c$

$= e^x - e^{-x} + c$

Example 2

Given $y = 5^x$, use the rules of logarithms to find x in terms of y.

Hence, find (i) $\frac{dx}{dy}$ (ii) $\frac{dy}{dx}$.

Use the result from (ii) to show that $\int 5^x\,dx = \frac{5^x}{\ln 5} + c$.

I don't like this question !

Integration

Constants of integration omitted.

$f(x)$	$\int f(x)dx$
$x^n, (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
e^{ax}	$\frac{1}{a}e^{ax}$
$a^x \quad (a > 0)$	$\frac{a^x}{\ln a}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\tan x$	$\ln\lvert\sec x\rvert$
$\frac{1}{\sqrt{a^2 - x^2}} \quad (a > 0)$	$\sin^{-1}\frac{x}{a}$
$\frac{1}{x^2 + a^2} \quad (a > 0)$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$

$\int 5^x dx = ?$

Integrate $\Rightarrow$ $= \frac{5^x}{\ln 5} + c$

Integrals of the form $\int \sin ax$ and $\int \cos ax$

In our study of differential calculus, we found the derivatives of $\sin x$, $\sin mx$, $\cos x$ and $\cos mx$.

We will now use these results to write down the standard integrals of the basic trigonometric functions.

(i) $\frac{d}{dx}\sin x = \cos \quad \Rightarrow \quad \int \cos x\,dx = \sin x + c$

(ii) $\frac{d}{dx}\sin mx = m\cos mx \quad \Rightarrow \quad \int \cos mx\,dx = \frac{\sin mx}{m} + c$

(iii) $\frac{d}{dx}\cos x = -\sin x \quad \Rightarrow \quad \int \sin x\,dx = -\cos x + c$

(iv) $\frac{d}{dx}\cos mx = -m\sin mx \quad \Rightarrow \quad \int \sin mx\,dx = -\frac{\cos mx}{m} + c$

Consider

$$y = \sin 2x$$

Differentiation

chain Rule

$$\frac{dy}{dx} = 2\cos 2x$$

$$\int \cos 2x\,dx = ?$$

Integration

$$= \frac{\sin 2x}{2} + c$$

Example 3

Find (i) $\int \cos 4x \, dx$ (ii) $\int \sin 3x \, dx$.

* learn these!

Integration

Constants of integration omitted.

$f(x)$	$\int f(x)dx$
$x^n, (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
e^{ax}	$\frac{1}{a}e^{ax}$
$a^x \quad (a>0)$	$\frac{a^x}{\ln a}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\tan x$	$\ln\lvert\sec x\rvert$
$\frac{1}{\sqrt{a^2-x^2}} \quad (a>0)$	$\sin^{-1}\frac{x}{a}$
$\frac{1}{x^2+a^2} \quad (a>0)$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$

(i) $\int \cos 4x \, dx$

$$= \frac{\sin 4x}{4} + c$$

(ii) $\int \sin 3x \, dx$

$$= -\frac{\cos 3x}{3} + c$$

Example 4

If $y = \sin 3x^2$, find $\frac{dy}{dx}$. Hence find $\int 6x \cos 3x^2 dx$

Differentiate

$\sin x \to \cos x$

$$\frac{dy}{dx} = \cos 3x^2 \cdot 6x$$

$$\frac{dy}{dx} = 6x \cos 3x^2$$

Integrate

$$y = \int 6x \cos 3x^2$$

$$y = \sin 3x^2 + c$$

Example 5

Let $h(x) = x\ln x,\ x \in R, x > 0.$

(i) Find $h'(x)$.

(ii) Hence, find $\int \ln x\,dx$.

* mistake in this example in the book

ignore

6. Find $\int 3(e^x - 4\sin 3x + 2)\,dx$.

3 separate integrals to consider

$$3\int e^x\,dx - 3\int 4\sin 3x\,dx + 3\int 2\,dx$$

$$= 3\int e^x\,dx - 12\int \sin 3x\,dx + 6\int dx$$

integrate

$$y = 3e^x + \frac{12\cos 3x}{3} + 6x + c$$

$$= 3e^x + 4\cos 3x + 6x + c$$

9. Given that $\frac{x+y}{z} = \frac{x}{z} + \frac{y}{z}$, use this identity to integrate each of the following:

(i) $\int \frac{e^{2x}+4}{e^x}\,dx$ (ii) $\int \frac{e^{x+2}+3}{e^x}\,dx$ (iii) $\int \frac{1+3e^x}{e^{2x}}\,dx$

(i) $\int \frac{e^{2x}+4}{e^x}\,dx$

Separate fraction

$= \int \left(\frac{e^{2x}}{e^x} + \frac{4}{e^x}\right) dx$

write in index form

$= \int \left(e^x + 4e^{-x}\right) dx$

Integrate

$\boxed{\begin{array}{l} f'(x) \to f(x) \\ e^{ax} \to \frac{1}{a}e^{ax} \end{array}}$

$= e^x + \frac{4e^{-x}}{-1} + c$

$= e^x - 4e^{-x} + c$