chapter 7 Algebra 3

Section 7.11 Problem-solving with exponential and log functions

PROJECT MATHS Text & Tests 6

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Example 1

The acidity of a substance is determined by the ion concentration formula $pH = -\log[H^+]$, where a pH of 7 is defined as neutral, <7 acidic, >7 alkaline. Determine the acidity of each of the following substances.

(a) Apple juice with a $[H^+]$ ion concentration of 0.0003.
(b) Ammonia with a $[H^+]$ ion concentration of 1.3×10^{-9}.

$pH = -\log(H^+)$

(a) $pH = -\log(0.0003) = 3.5$

(b) $pH = -\log(1.3 \times 10^{-9}) = 8.9$

Example 2

The loudness of a sound of intensity I is given by the formula $\text{dB} = 10\log\left(\frac{I}{I_o}\right)$, where dB is measured in decibels and I_o is the threshold intensity of hearing ($I_o = 1 \times 10^{-12}\,\text{Wm}^{-2}$).

(a) Find the loudness (in decibels) of a sound at the threshold of hearing.

(b) Given that prolonged exposure to sounds over 85 decibels can cause hearing damage, and that a gunshot from a .22 rifle has an intensity of $I = \boxed{2.5 \times 10^{13} I_o}$, should you wear ear protection when firing this gun?

$dB = 10\log\left(\frac{I}{I_o}\right)$

$I_o = 1\times10^{-12}\,Wm^{-2}$

a) $I = I_o$

$dB = 10\log\left(\frac{1\times10^{-12}}{1\times10^{-12}}\right) = 0$

b) $I = 2.5\times10^{13}\,I_o = (2.5\times10^{13})(1\times10^{-12}\,Wm^{-2})$

$I = 25\ Wm^{-2}$

$dB = 10\log\left(\frac{25}{1\times10^{-12}}\right) \simeq 134\ dB$

YES!

Example 3

How long would it take €5000 to increase in value to €6000, if invested in a credit union at a yearly compound interest rate of 2%?

$F = P(1+i)^t$

$\boxed{\frac{F}{P}} = \boxed{(1+i)}^{\boxed{t}}$

$t = \log_{\boxed{1+i}}\boxed{F/P}$

$P = 5000$
$F = 6000$
$i = 2\%$
$t = ?$

$t = \log_{1.02}\left(\frac{6000}{5000}\right) = 9.2$ years

Example 4

The population of red squirrels in a given region was estimated to be 5000 at the start of 2003. Assuming a rate of decrease of 5% per year, estimate the size of the population in 2013.

Depreciation

$F = P(1-i)^t$

$F = ?$
$P = 5000$
$i = 5\%$
$t = 10$

$F = 5000(0.95)^{10} = 2994$

Example 5

A certain type of bacteria is growing exponentially, where $y = A\,e^{bt}$ is the number of bacteria present after t (hours) and b is the growth constant. Under certain conditions, the bacteria doubles in population every 6.5 hours. If at the start of the experiment under these conditions there are 100 bacteria present, find (i) the growth constant b (ii) how many bacteria will be present after 2 days.

$y = Ae^{bt}$

Initially
$t = 0, \; y = 100$

$100 = Ae^{b(0)}$
$100 = A(1)$ $\Big\} \Rightarrow A = 100$

$\Rightarrow y = 100e^{bt}$

$t = 6.5, \; y = 200$

$200 = 100e^{b6.5}$

$2 = e^{6.5b} \Rightarrow 6.5b = \log_e 2 \approx 0.69$*

$b = 0.69/6.5 = 0.11 = b$

$\Rightarrow y = 100e^{0.11t}$

$y = ?, \; t = 24 \times 2 = 48$

$y = 100e^{0.11(48)} = 19636$

* Could have use more decimal places.

2. A biologist puts 100 bacteria into a controlled environment at the start of an experiment.
Six hours later, she returns and counts 450 bacteria in the colony.
Assuming exponential growth of the form $y = Ae^{bt}$ where b is the growth constant, find a value for b, correct to two decimal places.

$y = Ae^{bt}$

Initially
$t=0,\ y=450$

After 6 hours
$t=6,\ y=450$

$0 = Ae^{b(0)}$
$0 = A(1)$
$A = 100$

$\Rightarrow y = 100e^{bt}$

$450 = 100e^{6b}$

$\Rightarrow 6b = \log_e\left(\frac{450}{100}\right) = 1.504$

$b = \frac{1.504}{6}$

$b = 0.25$

$\Rightarrow y = 100e^{t/4}$

4. The loudness L (measured in dB) of a sound is given by the formula $L = 10\log_{10}\left(\frac{I}{I_o}\right)$,
where I_o is the threshold of hearing $(1 \times 10^{-12}\,\text{Wm}^{-2})$ and I the intensity of the sound.
(i) If thunder can have a range of loudness between 100–110 dB, what is the corresponding range of intensities in Wm^{-2}?
(ii) The threshold of pain is generally assumed to be $10\,\text{Wm}^{-2}$. Find in dB the loudness of a sound that starts to cause pain.

Watts per m^2
$= \text{Wm}^{-2}$

$L = 10\log\left(\frac{I}{I_o}\right)$ (i)

$L = 100$ dB $\quad I = ?$

$\boxed{\text{if } a = b^n \Leftrightarrow \log_b a = n}$ *NB

$L = 110$ dB $\quad I = ?$

$100 = 10\log\left(\frac{I}{1\times10^{-12}}\right)$

$\boxed{10} = \log_{\boxed{10}}\boxed{\left(\frac{I}{1\times10^{-12}}\right)}$

$10^{10} = \frac{I}{1\times10^{-12}}$

$I = 10^{10}(1\times10^{-12}) = 0.01\ \text{W/m}^2$

$110 = 10\log\left(\frac{I}{1\times10^{-12}}\right)$

$11 = \log_{10}\left(\frac{I}{1\times10^{-12}}\right)$

$10^{11} = I/1\times10^{-12} \rightarrow I = 10^{11}(1\times10^{-12}) = 0.1\ \text{W/m}^2$